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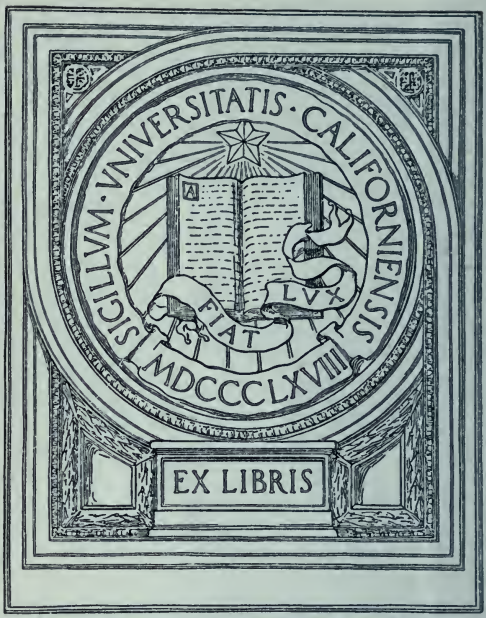
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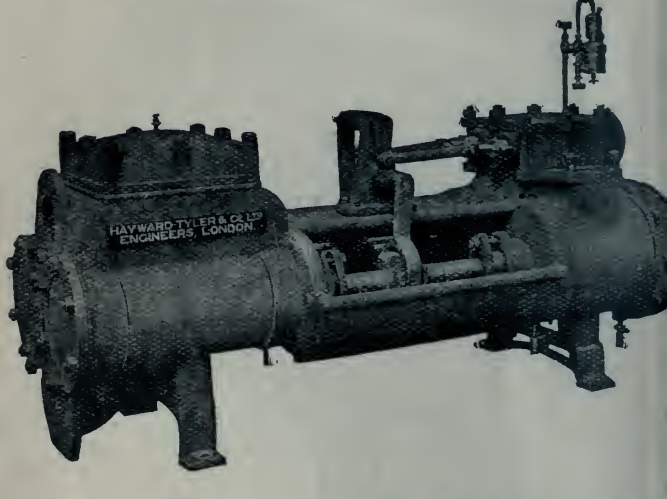

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THE

NAVAL ARCHITECT'S AND SHIPBUILDER'S

POCKET-BOOK

OF

Formulæ, Rules, and Tables

AND

MARINE ENGINEER'S AND SURVEYOR'S
HANDY BOOK OF REFERENCE

BY

CLEMENT MACKROW

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LATE LECTURER ON NAVAL ARCHITECTURE AT THE BOW AND BROMLEY INSTITUTE

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Eleventh Edition

THOROUGHLY REVISED

WITH A SECTION ON AERONAUTICS

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PREFACE

TO

THE ELEVENTH EDITION.

THE need of a new edition of this Pocket-book has arisen through the continual development of the science of Naval Architecture, and the tendency towards standardization and regulation of parts of the structure and equipment of ships. Very many changes have been introduced, and much of the book has been rewritten, but where possible its form has been left unaltered. Its object remains the same as that stated in the Preface to the original edition, viz. to condense into a compact form all data and formulæ that are ordinarily required by the Shipbuilder or Naval Architect.

Amongst the new matter inserted, it is believed that the section on Speed and Horse-power will be useful in enabling ships of ordinary form to be approximately powered from the data therein given; a brief description of modern methods of powering and determining forms suitable from a propulsive standpoint has also been included. The necessity for economizing weight where possible without diminution of strength has led to the sections on Strength of Materials, Riveted Joints, and Stresses in Ships being considerably extended. Information concerning British Standard Sections, Screws, Keys, etc., has also been added, by permission of the Engineering Standards Committee. Finally, two new sections on Aeronautical matters will be of service, not only to those engaged in that modern and rapidly

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developing branch of engineering, but also to Naval Architects on account of the kindred nature of the subjects, and of the direct application of many air data to questions relating to the resistance of bodies in water.

The remaining subjects treated, which were also included in previous editions, have now been brought completely up-to-date; the excision of obsolete data has enabled the new matter to be inserted without increase in the size of the book. The new tables of logarithms, etc., it is trusted, will be found of great practical convenience to those using them.

The scope and extent of the revision were arranged in the first place with the original author; although, owing to his death before the completion of the work, the absence of his advice and experience during the later stages has been felt and regretted, the reviser has had the benefit of securing great assistance from many sources during the preparation of the new edition. Among those who kindly contributed, the reviser is greatly indebted to Mr. A. W. Johns, the results of whose valued experience have been embodied in various parts of the book; the new sections 'Aerodynamics' and 'Aeronautics' are entirely due to him. Considerable aid in the treatment of Speed and Propellers has been rendered by Professor T. B. Abell, while Mr. E. F. Atkinson has supplied useful data concerning small craft and tugs. To these, and to many others to whom reference is made in the course of the book, the reviser tenders his cordial thanks. He also trusts that the numerous correspondents who have offered suggestions and pointed out errors in previous editions may be led to take the same kindly interest in the present revision.

L. W.

BARNES: *January* 1, 1916.

PREFACE

TO

THE FIRST EDITION.

THE OBJECT of this work is to supply the great want which has long been experienced by nearly all who are connected professionally with shipbuilding, of a Pocket-Book which should contain all the ordinary Formulæ, Rules, and Tables required when working out necessary calculations, which up to the present time, as far as the Author is aware, have never been collected and put into so convenient a form, but have remained scattered through a number of large works, entailing, even in referring to the most commonly used Formulæ, much waste of time and trouble. An effort has here been made to gather all this valuable material, and to condense it into as compact a form as possible, so that the Naval Architect or the Shipbuilder may always have ready to his hand reliable data from which he can solve the numerous problems which daily come before him. How far this object has been attained may best be judged by those who have felt the need of such a work.

Several elementary subjects have been treated more fully than may seem consistent with the character of the book. This, however, has been done for the benefit of those who have received a practical rather than a theoretical training, and to whom such a book as this would be but of small service were they not first enabled to

gather a few elementary principles, by which means they may learn to use and understand these Formulæ.

In justice to those authors whose works have been consulted, it must be added that most of the Rules and Formulæ here given are not original, although perhaps appearing in a new shape with a view to making them simpler.

There are many into whose hands this work will fall who are well able to criticise it, both as to the usefulness and the accuracy of the matter it contains. From such critics the Author invites any corrections or fresh material which may be useful for future editions.

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MACKROW AND WOOLLARD'S

POCKET BOOK

OF

FORMULÆ, RULES, AND TABLES

FOR

NAVAL ARCHITECTS AND SHIP-BUILDERS.

SIGNS AND SYMBOLS.

THE following are some of the signs and symbols commonly used in algebraical expressions:—

= This is the sign of equality. It denotes that the quantities so connected are equal to one another; thus, 3 feet = 1 yard.

+ This is the sign of addition, and signifies plus or more; thus, $4 + 3 = 7$.

− This is the sign of subtraction, and signifies minus or less; thus, $4 - 3 = 1$.

× This is the sign of multiplication, and signifies multiplied by or into; thus, $4 \times 3 = 12$.

÷ or / This is the sign of division, and signifies divided by; thus, $4 \div 2 = 2$ or $4/2 = 2$.

() {} [] These signs are called brackets, and denote that the quantities between them are to be treated as one quantity; thus, $5\{3(4+2) - 6(3-2)\} = 5(18-6) = 60$.

—— This sign is called the bar or vinculum, and is sometimes used instead of the brackets; thus, $\overline{3(4+2)-6(3-2)} \times 5 = 60$.

Letters are often used to shorten or simplify a formula. Thus, supposing we wish to express length × breadth × depth, we might put the initial letters only, thus, $l \times b \times d$, or, as is usual when algebraical symbols are employed, leave out the sign × between the factors and write the formula *l.b.d.*

When it is wished to express division in a simple form the divisor is written under the dividend; thus, $(x+y) \div z = \frac{x+y}{z}$

$:, ::, \therefore$ These are signs of proportion; the sign $:=$ is to, the sign $::=$ as; thus, $1:3::3:9$, 1 is to 3 as 3 is to 9.

$<$ This sign denotes less than; thus $2<4$ signifies 2 is less than 4.

$>$ This sign denotes more than; thus $4>2$ signifies 4 is more than 2.

$\because$ This sign signifies because.

$\therefore$ This sign signifies therefore. *Ex.*: $\because 9$ is the square of $3 \therefore 3$ is the root of 9.

$\sim$ This sign denotes difference, and is placed between two quantities when it is not known which is the greater; thus $(x \sim y)$ signifies the difference between x and y.

$^\circ, ', ''$ These signs are used to express certain angles in degrees, minutes, and seconds; thus 25 degrees 4 minutes 21 seconds would be expressed $25^\circ\ 4'\ 21''$.

Note.—The two latter signs are often used to express feet and inches; thus 2 feet 6 inches may be written $2'\ 6''$.

$\surd$ This sign is called the radical sign, and placed before a quantity indicates that some root of it is to be taken, and a small figure placed over the sign, called the exponent of the root, shows what root is to be extracted.

Thus $\surd a$ or $\sqrt{a}$ means the square root of a.

$\sqrt[3]{a}$ „ cube „

$\sqrt[4]{a}$ „ fourth „

$\frac{\sqrt{a}}{b}$ This denotes that the square root of a has to be taken and divided by b.

$\frac{b}{\sqrt{a}}$ This denotes that b has to be divided by the square root of a.

$\sqrt{\frac{a+b}{a+d}}$ This denotes that the square root of $a+b$ has to be divided by the square root of $a+d$. It may also be written thus, $\surd\frac{a+b}{a+d}$, or $\frac{\sqrt{a+b}}{\sqrt{a+d}}$.

$\propto$ This is another sign of proportion. *Ex.*: $a \propto b$; that is, a varies as or is proportional to b.

∞ This sign expresses infinity; that is, it denotes a quantity greater than any finite quantity.

0 This sign denotes a quantity infinitely small, nought.

$\angle$ This sign denotes an angle. *Ex.*: $\angle abc$ would be written, the angle abc.

$\llcorner$ This sign denotes a right angle.

$\perp$ This sign denotes a perpendicular; as, $ab \perp cd$, i.e. ab is perpendicular to cd.

$\triangle$ This sign denotes a triangle; thus, $\triangle abc$, i.e. the triangle abc.

$\parallel$ This sign denotes parallel to. *Ex.*: $ab \parallel cd$ would be written, ab is parallel to cd.

f or F These express a function; as, $a = f(x)$; that is, a is a function of x or depends on x.

$\int$ This is the sign of integration; that is, it indicates that the expression before which it is placed is to be integrated. When the expression has to be integrated twice or three times the sign is repeated (thus, $\iint, \iiint$); but if more than three times an index is placed above it (thus, $\int^n$).

D or d These are the signs of differentiation; an index placed above the sign (thus, d^4) indicates the result of the repetition of the process denoted by that sign.

Σ This sign (the Greek letter sigma) is used to denote that the algebraical sum of a quantity is to be taken. It is commonly used to indicate the sum of finite differences, just as the symbol $\int$ is used for indefinitely small differences.

g This sign is used to denote the acceleration due to gravity at any given latitude. Its value is about 32·2 in foot-second units and 981 in C.G.S. units.

π The Greek letter pi is invariably used to denote 3·14159; that is, the ratio borne by the diameter of a circle to its circumference.

e or ϵ This letter is generally used to denote 2·71828, which is the base of hyperbolic or Napierian logarithms.

$\lfloor n$ or $n!$ termed 'factorial n', where n is a positive integer, denotes the product of the series $n\ (n-1)\ (n-2)\ \ldots\ 2.1$. Thus, $\lfloor 3 = 3.2.1$ or 6; and $\lfloor 5 = 5.4.3.2.1 = 120$.

$\bowtie$ denotes the midship section or midship part of a vessel.

As the letters of the Greek alphabet are of constant recurrence in mathematical formulæ it has been deemed advisable to append the following table:—

A α Alpha.	I ι Iota.	P ρ Rho.
B β Beta.	K κ Kappa.	Σ σ s Sigma.
Γ γ Gamma.	Λ λ Lambda.	T τ Tau.
Δ δ Delta.	M μ Mu.	Υ υ Upsilon.
E ϵ Epsilon.	N ν Nu.	Φ ϕ Phi.
Z ζ Zeta.	Ξ ξ Xi.	X χ Chi.
H η Eta.	O o Omicron.	Ψ ψ Psi.
Θ θ Theta.	Π π Pi.	Ω ω Omega.

LOGARITHMS.

DEFINITION.—The logarithm of a number to a given base is the index of the power to which the base must be raised in order to become equal to the given number. Thus, if $a^x = N$, x is called the logarithm of N to base a.

The logarithms naturally occurring in analytical formulæ are to the base e, which is equal to 2·718 . . . or to the sum of the infinite series $1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots$; the values of the logarithms are obtained indirectly from the formula $\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ Such logarithms are termed Napierian or hyperbolic logarithms; their values are given in the table on pp. 700–4.

When used to shorten arithmetical work, 'common logarithms' are employed, having 10 as their base.

Note.—The logarithm of 1 to any base is zero.

TO CHANGE THE BASE OF A LOGARITHM.

RULE.—To obtain the logarithm of a number to base b from that to base a, multiply the latter logarithm by the logarithm of a to base b, or, equally, divide it by the logarithm of b to base a.

The logarithm of N to base a is denoted by $\log_a N$.

$$\therefore \log_b N = \log_a N \times \log_b a = \log_a N \div \log_a^{-1} b.$$

Since $\log_e 10 = 2{\cdot}303 \ldots = \frac{1}{{\cdot}4343 \ldots}$, the hyperbolic logarithm of a number is obtained by multiplying its common logarithm by 2·303 or $\frac{1}{{\cdot}4343}$

Note.—The integral part of a logarithm is termed its *characteristic*, and the decimal part its *mantissa*.

TO FIND THE LOGARITHM OF A NUMBER.

RULE.—The *characteristic* is one less than the number of digits in the integral part of the number; when there is no integral part, the characteristic is negative and is numerically one more than the number of cyphers between the decimal point and the first significant figure. In the latter case the minus sign is placed *over*, instead of before, the characteristic.

The *mantissa* is invariably positive; its value for numbers of three or less significant figures is directly obtained from the tables on pp. 700-4; for numbers having four significant

figures the tabular differences given in the columns on the right are employed thus—

Ex. 1.—Find log of 42·63.

log	42·60 =	1·6294
tab. diff.	3 =	3
log	42·63 =	1·6297

Ex. 2.—Find log of ·7897.

log	·7890 =	$\bar{1}$·8971
tab. diff.	7 =	4
log	·7897 =	$\bar{1}$·8975

Note.—The tabular difference is placed under the extreme right-hand figure or figures of the mantissa.

To find the Antilogarithm, or the Number corresponding to a given Logarithm.

Rule.—From the tables of antilogarithms, find the number corresponding to the given logarithm, using the tabular differences as before if four significant figures are required. If the characteristic is positive, the decimal point is so placed that the number of digits to the left is one more than the characteristic; if negative, the number of ciphers between the decimal point and the first significant figure is one less than the characteristic. For tables v. pp. 705-8.

Ex. 1.—Find the number whose logarithm is 5·8178.

antilog	8170 =	6561
tab. diff.	8 =	14
antilog	8178 =	6575

Number required is 657,500 to four significant figures.

Ex. 2.—Find the number whose logarithm is $\bar{3}$·1763

antilog	1760 =	1500
tab. diff.	3 =	1
antilog	1763 =	1501

Number required is ·001501.

To Multiply and Divide by Logarithms.

Rule.—Add together the logarithms of the numbers in the numerator, and those of the numbers in the denominator; subtract the latter sum from the former. The antilogarithm of the result is the number required.

Ex.: Evaluate $\frac{2}{3} \times \frac{17{\cdot}63}{35} \times \frac{2{\cdot}052}{{\cdot}008175}$

log 2	=	·3010	log 3	=	·4771
log 17·63	=	1·2462	log 35	=	1·5441
log 2·052	=	·3122	log ·008175	=	$\bar{3}$·9125
		1·8594			$\bar{1}$·9337
subtract		$\bar{1}$·9337			
antilog		$\bar{1}$·9257	=	84·28—the required result to four significant figures.	

Note.—It is advisable to perform the operations of addition, multiplication, etc., on the mantissa and characteristic separately.

Involution and Evolution by Logarithms.

Rule.—Multiply the logarithm of the number by the index of the power to which it is to be raised. The antilogarithm of the result is the number required.

Ex. 1.—Find the cube and cube root of ·9873.

log ·9873 = $\bar{1}$·9944	log ·9873 = 1·9944
Multiply by 3	$= -3 + 2·9944$
$\bar{1}$·9832	Divided by 3)
Antilog $\bar{1}$·9832 = ·9620 which is the cube of ·9873.	$\bar{1}$·9981
	Antilog $\bar{1}$·9981 = ·9956, which is the cube root of ·9873.

Ex. 2.—Evaluate $(20·4)^{1·83}$.

log 20·4 = 1·3096, say 1·310.
To multiply this by 1·83,
log 1·310 = ·1173
log 1·83 = ·2625

·3798

Antilog ·3798 = 2·397 ; antilog 2·397 = 242·5, the required result.

Accuracy of Numerical Calculations.

In general, the accuracy of the result of a numerical calculation is the same as that of the factor liable to the greatest proportional error. Exceptional cases arise, viz., (*a*) when two nearly equal numbers are subtracted the percentage error in the result is usually greater than that in either of the numbers ; (*b*) when a large number of similar quantities, such as the ordinates in a displacement sheet, are added, the individual errors of measurement tend to neutralize, and the accuracy of the result is usually greater than that of its component factors ; (*c*) the percentage error in the n^{th} power of a number is n times that of a number ; thus in the cube the error is trebled, but in the cube root it is divided by three. Subject to these qualifications a considerable saving in the numerical labour of a calculation may be effected by limiting the number of significant figures at each stage to that appropriate to the accuracy of the result.

In calculations affecting the weight, buoyancy, stability, speed, strength, etc., of ships, a proportional error of at least 0·1 per cent, i.e. one in a thousand, may generally be expected ; three or, at most, four significant figures are sufficient in such cases, any additional figures being meaningless and redundant.

The slide rule, which mechanically performs the operations of multiplication, division, evolution, etc., by the aid (virtually) of three-place logarithms, is usually sufficiently accurate for the majority of such calculations ; tables of logarithms, trigonometrical functions, etc., to four (or at most five) places of decimals are sufficient to perform any calculations in which rather greater accuracy is desired and can be obtained.

TRIGONOMETRY.

The *complement* of an angle is its defect from a right angle ; thus if A denote the number of degrees contained in any angle, $90° - A$ is the number of degrees contained in the complement of that angle.

The *supplement* of an angle is its defect from two right angles ; thus $180° - A$ is the number of degrees contained in the supplement of that angle.

TRIGONOMETRICAL RATIOS.

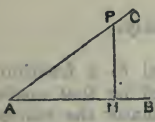


FIG. 1.

The trigonometrical ratios of an angle are defined as follows :—Let BAC (fig. 1) be any angle ; take any point in either of the containing sides and from it draw a perpendicular to the other side ; let P be the point in the side AC, and PM perpendicular to AB ; let A denote the angle BAC. Then—

$$\text{sine A} = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{\text{PM}}{\text{AP}}$$

$$\text{co-sine A} = \frac{\text{base}}{\text{hypotenuse}} = \frac{\text{AM}}{\text{AP}}$$

$$\text{tangent A} = \frac{\text{perpendicular}}{\text{base}} = \frac{\text{PM}}{\text{AM}}$$

$$\text{co-tangent A} = \frac{\text{base}}{\text{perpendicular}} = \frac{\text{AM}}{\text{PM}}$$

$$\text{secant A} = \frac{\text{hypotenuse}}{\text{base}} = \frac{\text{AP}}{\text{AM}}$$

$$\text{co-secant A} = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{\text{AP}}{\text{PM}}$$

$$\text{versed sine A} = 1 - \cos \text{A}$$
$$\text{co-versed sine A} = 1 - \sin \text{A}.$$

These ratios depend only on the angle, and are independent of the position of the point P.

Measurement of Angles.

There are three modes of measuring angles, viz.—
1st. The sexagesimal or English method.
2nd. The centesimal or French method.
3rd. The circular measure.

The sexagesimal method and the circular measure only will be dealt with here.

The Sexagesimal Method.—In this method a right angle is supposed to be divided into 90 equal parts, each of which parts is termed a degree ; each degree is divided into 60 equal parts called minutes, and each minute is divided into 60 equal parts called seconds. One degree 16 minutes 15 seconds or $1° 16' 15''$, is therefore equal to $1 + \frac{16}{60} + \frac{15}{3600}$ or 1·271 degrees.

The Circular Measure.—The unit of circular measure is an angle which is subtended at the centre of a circle by an arc equal to the radius of that circle. It is called a radian. Such an angle is equal to

$$\frac{2 \text{ right angles}}{\pi} = \frac{180°}{3{\cdot}1416} = 57{\cdot}3° \text{ nearly.}$$

The circular measure of an angle is equal to a fraction which has for its numerator the arc subtended by that angle at the centre of any circle, and for its denominator the radius of that circle.

Since the circumference of any circle is 2π times the radius, four right angles are equal to 2π radians. Consequently one right angle is equal to $\frac{\pi}{2}$ radians.

Approximate values of π are 3·1416 and $\frac{22}{7}$ and $\frac{355}{113}$

To find the circular measure of any angle expressed in degrees, minutes, and seconds.

Rule.—Multiply the measure of the angle in degrees by π, and divide by 180.

Ex.: Express $1° 16' 15''$ or 1·271° in circular measure.

$$\frac{1{\cdot}271 \times \pi}{180} = {\cdot}0222 \text{ circ. meas.}$$

To find the measure of any angle in degrees, minutes, and seconds, the circular measure being given.

Rule.—Multiply the circular measure of the angle by 180, and divide by π.

Ex. 1.—Express in degrees, etc., an angle the circular measure of which is $\frac{2\pi}{3}$

$$\frac{2\pi \times 180}{3 \times \pi} = 120°.$$

Tables giving the circular measure of angles are on pp. 639-41.

GENERAL FORMULÆ.

$$\sin^2\theta + \cos^2\theta = 1. \qquad \sec^2\theta = 1 + \tan^2\theta.$$

$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta.$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

$$\sin A + \sin B = 2 \sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos B - \cos A = 2 \sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A. \qquad \sin 3A = 3 \sin A - 4 \sin^3 A.$$

$$\cos 2A = \cos^2 A - \sin^2 A. \qquad \cos 3A = 4 \cos^3 A - 3 \cos A.$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}} \qquad \cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}$$

If $t = \tan\frac{A}{2}$, $\sin A = \frac{2t}{1+t^2}$; $\cos A = \frac{1-t^2}{1+t^2}$; $\tan A = \frac{2t}{1-t^2}$

And when A, B, C are the three angles of a triangle,

$$A + B + C = \pi \text{ radians or two right angles;}$$

and $$\sin(A + B) = \sin(\pi - C) = \sin C.$$

When A is any angle,

$$\sin(-A) = -\sin A. \qquad \cos(-A) = \cos A. \qquad \tan(-A) = -\tan A.$$

$$\sin(90° - A) = \cos A. \qquad \cos(90° - A) = \sin A. \qquad \tan(90° - A) = \cot A.$$

$$\sin(90° + A) = \cos A. \qquad \cos(90° + A) = -\sin A. \qquad \tan(90° + A) = -\cot A.$$

$$\sin(180° - A) = \sin A. \qquad \cos(180° - A) = -\cos A. \qquad \tan(180° - A) = -\tan A.$$

$$\sin(180° + A) = -\sin A. \qquad \cos(180° + A) = -\cos A. \qquad \tan(180° + A) = \tan A.$$

The algebraic formulæ for the sine and cosine are—

$$\sin A = \frac{e^{A\sqrt{-1}} - e^{-A\sqrt{-1}}}{2} = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \ldots$$

$$\cos A = \frac{e^{A\sqrt{-1}} + e^{-A\sqrt{-1}}}{2} = 1 - \frac{A^2}{2!} + \frac{A^4}{4!} - \ldots$$

where A is in circular measure.

Where A is small, $\sin A = \tan A = A$; $\cos A = 1 - \frac{A^2}{2}$; $\sec A = 1 + \frac{A^2}{2}$

Tables of the trigonometrical functions are given on pp. 716–19.

Inverse Functions.

If $\sin \alpha = x$, then $\alpha = \sin^{-1} x$.
If $\cos \alpha = y$, then $\alpha = \cos^{-1} y$.
And so on.

Note.—$\sin^{-1} x$ is read 'inverse sine x', etc.

Logarithmic Functions.

The logarithms of the sines, cosines, etc., are denoted log sin, log cos, etc., and their values are given on pp. 720-3. For convenience the characteristic is in each case increased by the number 10.

Properties of Triangles.

Fig. 2.

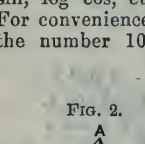


Fig. 3.

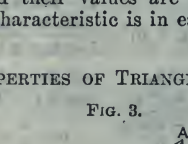
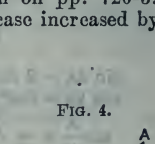


Fig. 4.



Note.—The sides opposite the angles A, B, C respectively will be denoted by the letters *a*, *b*, *c*. The angle BDA in figs. 2 and 3 is a right angle.

In fig. 2, where B and C are acute angles, we have—

$$\sin B = \frac{AD}{AB} = \frac{AD}{c}$$

$$\sin C = \frac{AD}{AC} = \frac{AD}{b}$$

$$\therefore \frac{\sin B}{\sin C} = \frac{AD}{c} \div \frac{AD}{b} = \frac{b}{c}$$

In fig. 3, where C is an obtuse angle, and in fig. 4, where C is a right angle, the proof is similar.

And therefore in any triangle $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Also $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}; \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}; \quad \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

where $2s = a + b + c$.

$$a = b \cos C + c \cos B; \qquad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

Area of triangle $= \frac{bc}{2} \sin A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$

Solution of Triangles.

Every triangle has six elements—three sides and three angles. If any three of these be given (provided they be not the three angles) the triangle can be completely determined.

Right-angled Triangles.

Let C be the right angle, and therefore c the hypotenuse.

(i.) Given hypotenuse (c) and one side (a).

$$b = \sqrt{c^2 - a^2}, \tan B = \frac{b}{a}, \text{ and } A = 90° - B.$$

(ii.) Given the two sides (a and b).

$$c = \sqrt{a^2 + b^2}, \tan B = \frac{b}{a}, \text{ and } A = 90° - B.$$

(iii.) Given an angle (B) and one of the sides (a).

$$b = a \tan B, \quad c = a \sec B.$$

(iv.) Given an angle (B) and the hypotenuse (c).

$$a = c \cos B, \quad b = c \sin B, \quad A = 90° - B.$$

Any Triangles.

(i.) Given the three sides, a, b, and c.

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}},$$

where $C = 180° - A - B$,

$2s = a + b + c$.

(ii.) Given two sides, b and c, and the included angle A.

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}. \qquad \frac{B+C}{2}=90^\circ-\frac{A}{2}.$$

From $\frac{B-C}{2}$ and $\frac{B+C}{2}$ we can get B and C; and $a=b\frac{\sin A}{\sin B}$.

(iii.) Given two sides, b and c, and the angle B opposite to one of them.

$\sin C=\frac{c}{b}\sin B$. We thus obtain C; and $A=(180-B-C)$.

Also $$a=b\frac{\sin A}{\sin B}.$$

As there are generally two angles between 0° and 180° whose sine is $\frac{c}{b}\sin B$, two values of C are often admissible, and sometimes two triangles can be constructed.

(iv.) Given one side and two angles, a, B, and C.

$$A=180^\circ-B-C;\ b=a\frac{\sin B}{\sin A};\ c=a\frac{\sin C}{\sin A}.$$

(v.) When the three angles only are given, the absolute magnitude of the sides cannot be determined, but their ratios are given by $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$.

TABLE GIVING THE SIGNS AND VALUES OF THE TRIGONOMETRICAL RATIOS FOR CERTAIN ANGLES.

Ratios	0°	Signs	30°	Signs	45°	Signs	60°	Signs	90°	Signs	120°
Sine	0	+	$\frac{1}{2}$	+	$\frac{1}{\sqrt{2}}$	+	$\frac{\sqrt{3}}{2}$	+	1	+	$\frac{\sqrt{3}}{2}$
Co-sine	1	+	$\frac{\sqrt{3}}{2}$	+	$\frac{1}{\sqrt{2}}$	+	$\frac{1}{2}$	+	0	−	$\frac{1}{2}$
Tangent	0	+	$\frac{1}{\sqrt{3}}$	+	1	+	$\sqrt{3}$	+	∞	−	$\sqrt{3}$
Co-tangent	∞	+	$\sqrt{3}$	+	1	+	$\frac{1}{\sqrt{3}}$	+	0	−	$\frac{1}{\sqrt{3}}$
Secant	1	+	$\frac{2}{\sqrt{3}}$	+	$\sqrt{2}$	+	2	+	∞	−	2
Co-secant	∞	+	2	+	$\sqrt{2}$	+	$\frac{2}{\sqrt{3}}$	+	1	+	$\frac{2}{\sqrt{3}}$

Ratios	Signs	135°	Signs	150°	Signs	180°	Signs	270°	Signs	360°
Sine	+	$\frac{1}{\sqrt{2}}$	+	$\frac{1}{2}$	+	0	−	1	−	0
Co-sine	−	$\frac{1}{\sqrt{2}}$	−	$\frac{\sqrt{3}}{2}$	−	1	−	0	+	1
Tangent	−	1	−	$\frac{1}{\sqrt{3}}$	+	0	+	∞	−	0
Co-tangent	−	1	−	$\sqrt{3}$	+	∞	+	0	−	∞
Secant	−	$\sqrt{2}$	−	$\frac{2}{\sqrt{3}}$	−	1	−	∞	+	1
Co-secant	+	$\sqrt{2}$	+	2	+	∞	−	1	−	∞

Hyperbolic Functions.

The hyperbolic functions are used in connection with the catenary ; they are six in number, and are represented by affixing h to the symbols of the trigonometrical functions. They are determined by the following formulæ :—

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \ldots$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \ldots$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{cosech} x = \frac{1}{\sinh x}.$$

Note.—All formulæ connecting sin, cos, and tan can be converted into the corresponding formulæ for sinh, cosh, tanh by changing sin x to $\sqrt{-1}$ sinh x, cos x to cosh x, and tan x to $\sqrt{-1}$ tanh x ; thus $\cosh^2 x - \sinh^2 x = 1$; $\operatorname{sech}^2 x + \tanh^2 x = 1$; etc.

The values of sinh x, cosh x, e^x, and e^{-x} are given in the tables on pp. 708–10.

CURVES.

CONIC SECTIONS.

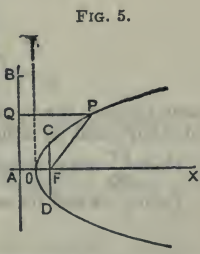
Definition.—The locus of a point which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed straight line is called a conic section.

The fixed point is called the focus, the constant ratio the eccentricity, and the fixed straight line the directrix.

The straight line passing through the focus and perpendicular to the directrix is called the axis.

Parabola.

Fig. 5.



The conic section is called a parabola when the eccentricity is equal to unity.

In fig. 5, F is the focus, AB the directrix, AX the axis, O the intersection of the curve with the axis, OY a line perpendicular to AX, and P any point on the curve ; then PQ = PF.

The equation of the curve with OY and OX as axes is

$y^2 = 4ax$, where AO = OF = a.

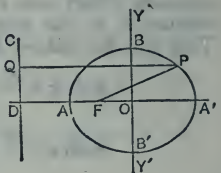
A parabola may also be defined as the section of a cone cut by a plane parallel to one of the slant sides.

ELLIPSE.

The conic section is called an ellipse when the eccentricity is less than unity.

In fig. 6 CD is the directrix, F the focus, AA′ the major axis, O the middle point of AA′, and BB′ the minor axis, down through O, perpendicular to the axis, and P any point on the curve so that $\frac{PF}{PQ}$ = the eccentricity e. The equation to the curve with OA′, OY as axes is—

FIG. 6.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Also OA = OA′ = a; OB = OB′ = b; $a^2 - b^2 = a^2 e^2$; OF = ae; OD = $\frac{a}{e}$

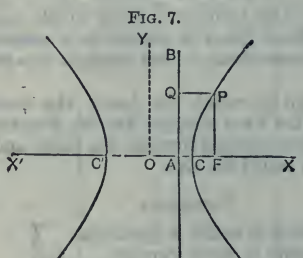
An ellipse may also be defined as the intersection of a cone by a plane passing through its slant sides, but not perpendicular to the axis.

HYPERBOLA.

The conic section is called a hyperbola when the eccentricity is greater than unity.

In fig. 7 AB is the directrix, F the focus, XX′ the axis, CC′ the points where the curve intersects the axis, OY a line

FIG. 7.



drawn through the middle point of CC′ perpendicular to the axis, and P any point on either branch of the curve.

Then $\frac{PF}{PQ}$ = the eccentricity e.

Taking OX and OY as axes the equation to the curve is—

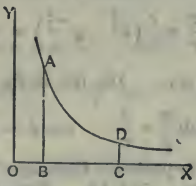
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Also OC = OC′ = a; $a^2 + b^2 = a^2 e^2$; OF = ae; OA = $\frac{a}{e}$

If the sides of a cone be produced beyond the vertex so as to form a second cone with the same axis as the first, and these two cones be cut by a plane, the section will be a hyperbola.

If b be made equal to a in the above equation, it becomes $x^2 - y^2 = a^2$, which is a *rectangular* hyperbola. By turning

FIG. 8.



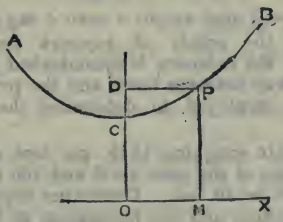
the axis through an angle of 45°, the equation becomes of the form $xy = c^2$ (fig. 8), where $c^2 = \frac{1}{2}a^2$

CATENARY.

(See pp. 27 and 28 for method of construction.)

If a uniform chain be freely suspended from two points, A and B, the curve in which it will hang is termed a common catenary ; the parameter OC is equal to the length of a piece of the chain whose weight is equal to the tension at the lowest point C in the curve.

FIG. 9.



The directrix OX is a horizontal line drawn through the extremity O of the parameter.

The tension at any point P in the curve is equal to the weight of a piece of the chain whose length is equal to the ordinate PM.

Equations to the Catenary (see fig. 9).

Take OX (horizontal) and OD (vertical through C the lowest point) as axes.

x = abscissa OM. y = ordinate PM. c = parameter OO. s = length CP of chain. w = weight of chain per linear unit run. T = tension at P. ϕ = angle to horizontal of chain at P. e = base of hyperbolic logarithms = 2·718 . . .

$$y = c\cosh\frac{x}{c} = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) = \sqrt{(c^2 + s^2)}.$$

$$s = c\sinh\frac{x}{c} = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) = \sqrt{(y^2 - c^2)}.$$

$$T = wc\cosh\frac{x}{c} = wy. \qquad \text{Dip (DC in fig.)} = c\left(\cosh\frac{x}{c} - 1\right).$$

$$\tan\phi = \sinh\frac{x}{c} = \frac{s}{c} \qquad \sec\phi = \cosh\frac{x}{c} = \frac{y}{c}$$

The values of the hyperbolic functions (sinh x, cosh x, etc.) are tabulated on pp. 708-10. Examples showing their application to the catenary are given below.

Approximate Equations for flat piece of chain, nearly horizontal.

$$\delta = \text{dip DC} = y - c\ ; \quad x = \tfrac{1}{2}\text{ span}.$$

$$\delta = \frac{1}{2}\frac{x^2}{c} + \frac{1}{24}\frac{x^4}{c^3} = \sqrt{\left(\frac{3}{2}x(s - x)\right)}$$

$$c = \frac{1}{2}\frac{x^2}{\delta} + \frac{1}{6}\delta = \frac{s^2 - \delta^2}{2\delta} = x\sqrt{\frac{x}{6(s - x)}}$$

$$s = x + \frac{1}{6}\frac{x^3}{c^2} = x + \frac{2}{3}\frac{\delta^2}{x}$$

Tension * = $\frac{1}{8}$ total weight × span ÷ sag at centre.

Note.—When the points of support are in the same horizontal plane, the catenary is symmetrical about a vertical line passing midway between them, and the preceding formulæ can be directly employed to determine the particulars of the curve.

Ex. 1.—A chain weighing 15 lb. per foot run is suspended between two points at the same level and 100 feet apart. The dip is observed to be 40 feet. Determine the length of chain, the maximum tension, and the inclination at the supports.

$$\text{Dip} = c\left(\cosh\frac{x}{c} - 1\right). \quad \text{Here dip} = 40\ ;\ x = \frac{100}{2} = 50.$$

Hence $40 = c\left(\cosh\frac{50}{c} - 1\right)$. By trial, from the tables (p. 709), $c = 36$ approximately.

$$\text{Length of chain} = 2s = 2c\sinh\frac{x}{c} = 135 \text{ feet approximately.}$$

* On substituting 'pressure' for 'weight', this is applicable to a rope or net under uniform pressure when sag is moderate.

Maximum tension occurs at supports and is given by—

$T = wc \cosh \frac{x}{c} = 1150$ lb.

Angle at supports $= \phi = \sec^{-1} \cosh \frac{x}{c} = 62°$.

Ex. 2.—The chain in the preceding example is tightened until the length suspended is reduced to 120 feet. To determine the dip—

$s = c \sinh \frac{x}{c}$. Here $s = 60$; $x = 50$.

Hence $60 = c \sinh \frac{50}{c}$. By trial, from the tables, $c = 47$ approximately.

$\text{Dip} = c\left(\cosh \frac{x}{c} - 1\right) = 29{\cdot}2$ ft.

Ex. 3.—If the chain in example 2 is tightened further until the dip is reduced to 9 feet, determine the length, and the tension.

Using the approximate formulæ for a flat chain—

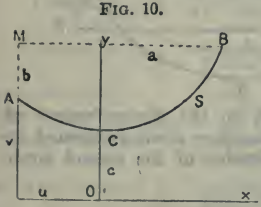
$\text{Length} = 2s = 2x + \frac{4}{3}\frac{\delta^2}{x}$

Here $x = 50$, $\delta = 9$. Hence length $= 102$ feet approximately.

$\text{Tension} = wc = w\left(\frac{1}{2}\frac{x^2}{\delta} + \frac{\delta}{6}\right) = 2{,}105$ lb.

Formulæ for the Catenary between two points not in the same horizontal plane.

FIG. 10.



Take axes as before (fig. 10), let A, B be the points of support

s = total length of chain ACB.

b = vertical distance AM between A and B.

a = horizontal distance MB between A and B.

v = height of A above axis OX

$\sqrt{s^2 - b^2} = 2c \sinh \frac{a}{2c}$ $\qquad s = 2c \sinh \frac{a}{2c} \cosh \frac{a - 2u}{2c}$

$b = 2c \sinh \frac{a}{2c} \sinh \frac{a - 2u}{2c}$ $\qquad v = c \cosh \frac{u}{c}$

Ex. 1.—A chain of length 100 feet is suspended between supports distant 50 feet horizontally and 60 feet vertically. Determine the position of its lowest point, and the maximum tension (weight 10 lb. per foot run).

Here S = 100; $a = 50$; $b = 60$.

By trial from the above formulæ, using the tables.

$$c = 14{\cdot}2\,;\quad u = 15{\cdot}2\,;\quad v = 23.$$

Maximum tension (at B) $= w\,c \sinh \dfrac{a-u}{c} = 820$ lb.

Note.—If a is negative, the lowest point of the catenary occurs outside the point of support A; in that case no part of the chain is horizontal.

Cycloidal Curves.

Definition.—If a circle be made to roll without slipping on a straight line, the locus of a point P (fig. 11) on its circumference is the cycloid MNM′, and that of any point Q inside the circle is the trochoid RSR′.

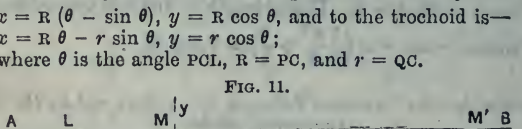
The cycloid meets the straight line AB at a series of cusps MM′ . . . , corresponding to the positions when the point P is vertically above the centre C of the rolling circle; on the trochoid these become 'crests' similar to those in the section of a wave-surface, which this curve is found closely to resemble; intermediate between M and M′ is the 'trough' S which has a smaller curvature than the 'crest'. With co-ordinates Ox, Oy, as shown, the equations to the cycloid is—

$x = \text{R}\,(\theta - \sin\theta)$, $y = \text{R}\cos\theta$, and to the trochoid is—

$x = \text{R}\,\theta - r\sin\theta$, $y = r\cos\theta$;

where θ is the angle PCL, R = PC, and r = QC.

Fig. 11.



The curve described by a point on the circumference of a circle rolling on the exterior of another circle is termed an epicycloid; when rolling on the interior of the second circle it is termed a hypocycloid.

Evolutes and Involutes.

Definition.—If a curve be drawn passing through the centres of curvature at various points at a curve, the new curve is said to be the evolute of the original curve; conversely the original curve is termed the involute of the derived curve.

The involute of a curve is also derived by wrapping a thread around the circumference of the curve; the path described by a point on the thread as it is wound or unwound is the involute.

DIFFERENTIAL AND INTEGRAL CALCULUS.

Differential Calculus.

Definition.—A quantity is said to be a function of another when its value depends upon the value of the other.

Thus, x^2, $\sin x$, e^x are functions of x; xy is a function x and of y; and so on. A function of x is denoted by $f(x)$. The differential coefficient of a function (y) with respect to a variable (x) is the rate of increase of y corresponding to an indefinitely small increment of x. It is denoted by $\frac{dy}{dx}$ or $f'(x)$.

Thus, the speed of a ship is the differential coefficient of the distance travelled with respect to the time occupied.

Values of the differential coefficients with respect to x of some functions of x are given in the table below :—

y or $f(x)$	$\frac{dy}{dx}$ or $f'(x)$	y or $f(x)$	$\frac{dy}{dx}$ or $f'(x)$
x^n	$n x^{n-1}$	a^x	$a^x \log_e a$
uv	$u\frac{dv}{dx}+v\frac{du}{dx}$	$\frac{u}{v}$	$\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
$\sin x$	$\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$-\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$\sec^2 x$	$\tanh x$	$\operatorname{sech}^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$	$\coth x$	$-\operatorname{cosech}^2 x$
$\sec x$	$\sec x \cdot \tan x$	$\operatorname{sech} x$	$-\operatorname{sech} x \cdot \tanh x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cdot \cot x$	$\operatorname{cosech} x$	$-\operatorname{cosech} x \cdot \coth x$
$f(u)$	$\frac{du}{dx}\times\frac{df(u)}{du}$	$\log_e x$	$\frac{1}{x}$
		$\sinh^{-1} x$ or $\log(x+\sqrt{x^2+1})$	$\frac{1}{\sqrt{x^2+1}}$
$\sin^{-1}x$ or $-\cos^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$		
		$\cosh^{-1} x$ or $\log(x+\sqrt{x^2-1})$	$\frac{1}{\sqrt{x^2-1}}$
$\tan^{-1}x$ or $-\cot^{-1}x$	$\frac{1}{1+x^2}$		
		$\tanh^{-1} x$ or $\frac{1}{2}\log\frac{1+x}{1-x}$	$\frac{1}{1-x^2}$
$\sec^{-1}x$ or $-\operatorname{cosec}^{-1}x$	$\frac{1}{x\sqrt{x^2-1}}$		
		$\coth^{-1} x$ or $\frac{1}{2}\log\frac{x+1}{x-1}$	$\frac{1}{1-x^2}$
$\operatorname{vers}^{-1}x$ or $-\operatorname{covers}^{-1}x$	$\frac{1}{\sqrt{2x-x^2}}$		

Note.—The differential coefficient of $\frac{dy}{dx}$ with respect to x is denoted $\frac{d^2y}{dx^2}$, and is termed the second differential coefficient of y.

Application to Curves.

The equation of the tangent at a point (x, y) on a curve is $Y - y = \frac{dy}{dx}(X - x)$.

That of the normal to the curve is $X - x + (Y - y)\frac{dy}{dx} = 0$.

The angle ϕ made by the tangent with the x axis, and the perimeter s measured from any fixed point on a curve are connected by—

$$\tan\varphi=\frac{dy}{dx};\ \sin\phi=\frac{dy}{ds};\ \cos\phi=\frac{dx}{ds};\ \left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2=1.$$

The radius of curvature (ρ) at a point is given by—

$$\rho=\pm\frac{ds}{d\phi}=\pm\frac{\left(\frac{ds}{dx}\right)^3}{\frac{d^2y}{dx^2}}=\pm\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}=\pm\frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}$$

When the curve is tangential to the x axis at the origin, then ρ is equal to $\frac{x^2}{2y}$ when x and y are very small.

Integral Calculus.

Definition.—If y is the differential coefficient of a function z with respect to x, then, conversely, z is termed the integral of y with respect to x, being denoted $\int y\,dx$.

A table of integrals frequently required is appended.

y or $\frac{dz}{dx}$	z or $\int y\,.\,dx$	y or $\frac{dz}{dx}$	z or $\int y\,.\,dx$
x^n	$\frac{x^{n+1}}{n+1}$	$\frac{1}{x}$	$\log x$
a^x	$a^x/\log_e a$	e^x	e^x
$\cos x$	$\sin x$	$\sin x$	$-\cos x$
$\cot x$	$\log\sin x$	$\tan x$	$\log\sec x$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\frac{1}{x\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{2ax-x^2}}$	$\mathrm{vers}^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\frac{x}{a}$
$\sqrt{a^2-x^2}$	$\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$	$\sqrt{a^2+x^2}$	$\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}$
$\sqrt{x^2-a^2}$	$\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}$	$\mathrm{cosec}\,x$	$\log\tan\frac{x}{2}$
$\sec x$	$\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\log\frac{x-a}{x+a}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\log\frac{a+x}{a-x}$	$u\,.\,v$	$u\int v\,.\,dx-\int\left(\frac{du}{dx}.\int v\,.\,dx\right)dx$

Definite Integrals.—On evaluating the integral $\int y \,.\, dx$. if two constant quantities a and b are separately substituted for x, and the former result substracted from the latter, the difference is termed a definite integral between limits a and b, being denoted by $\int_a^b y \,.\, dx$. Thus $\int x^3 \,.\, dx = \frac{x^4}{4}$, and $\int_2^3 x^3 \, dx = \frac{81-16}{4} = 16\frac{1}{4}$.

The following definite integrals are of frequent occurrence:—

$$\int_0^{\frac{\pi}{2}} \sin\theta \,.\, d\theta = 1 \qquad \int_0^{\frac{\pi}{2}} \cos\theta \,.\, d\theta = 1$$

$$\int_0^{\frac{\pi}{2}} \sin^2\theta \,.\, d\theta = \frac{\pi}{4} \qquad \int_0^{\frac{\pi}{2}} \cos^2\theta \,.\, d\theta = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^n\theta \,.\, d\theta = \int_0^{\frac{\pi}{2}} \cos^n\theta \,.\, d\theta = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdot \cdot \frac{2}{3}$$ when n is an odd integer.

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdot \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$ when n is an even integer.

$$\int_0^{\frac{\pi}{2}} \sin^m\theta \,.\, \cos^n\theta \,.\, d\theta$$ (m and n being integral)

$$= \frac{(m-1)(m-3) \ldots (n-1)(n-3) \ldots}{(m+n)(m+n-2)(m+n-4) \ldots}$$ when either m or n is odd.

$$= \frac{(m-1)(m-3) \ldots (n-1)(n-3) \ldots}{(m+n)(m+n-2)(m+n \quad 4) \ldots} \cdot \frac{\pi}{2}$$ when m and n are both even.

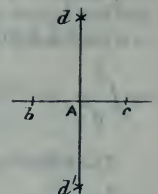
$$\int_0^{\infty} e^{-x^2} \,.\, dx = \tfrac{1}{2}\sqrt{\pi}.$$

PRACTICAL GEOMETRY.

1. *From any given point in a straight line to erect a perpendicular.* (Fig. 12.)

On each side of the point A in the line from which the perpendicular is to be erected set off equal distances A*b*, A*c*; and from *b* and *c* as centres, with any radius greater than A*b* or A*c*, describe arcs cutting each other at *d*, *d'*; a line drawn through *dd'* will pass through the point A, and A*d* will be perpendicular to *bc*.

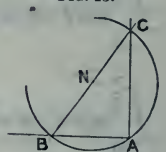
Fig. 12.



2. *To erect a perpendicular at or near the end of a line.* (Fig. 13.)

With any convenient radius, and at any distance from the given line AB, describe an arc, as BAC, cutting the given point in A; through the centre of the circle N draw the line BNC: a line drawn from the point A, cutting the intersection at C, will be the required perpendicular.

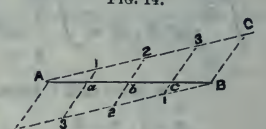
Fig. 13.



3. *To divide a line into any number of equal parts.* (Fig. 14.)

Let AB be the given straight line to be divided into a number of equal parts; through the points A and B draw two parallel lines AC and DB, forming any convenient angle with AB; upon AC and DB set off the number of equal parts required, as A–1, 1–2, &c., B–1, 1–2, &c.; join A and D, 1 and 3, 2 and 2, 3 and 1, C and B, cutting AB in *a*, *b*, and *c*, which will thus be divided into four equal parts.

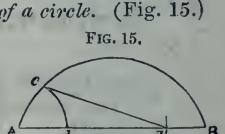
Fig. 14.



4. *To find the length of any given arc of a circle.* (Fig. 15.)

With the radius A*d*, equal to one-fourth of the length of the chord of the arc AB, and from A as a centre, cut the arc in *c*; also from B as a centre with the same radius cut the chord in *b*; draw the line *cb*, and twice the length of the line *cb* is the length of the arc nearly.

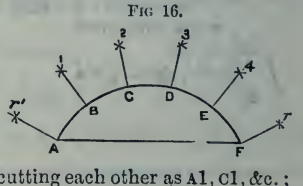
Fig. 15.



5. *To draw from or to the circumference of a circle lines tending towards the centre, when the centre is inaccessible.* (Fig. 16.)

Divide the given portion of the circumference into the desired number of parts; then with any radius greater than the distance of two parts, describe arcs cutting each other as A1, C1, &c.;

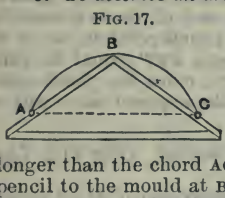
Fig 16.



draw the lines B1, C2, etc., which will lead to the centre, as required. *To draw the end lines* Ar′, Fr, from B and E with the same radii as before describe the arc r', r, and with the radius B1. from A as centre, cut the former arcs at r', r; lines then drawn from Ar′ and Fr will tend towards the centre, as required.

6. *To describe an arc of a circle of large radius.* (Fig. 17.)

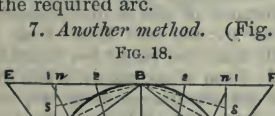
FIG. 17.



Let A, B, C be the three points through which the arc is to be drawn; join BA and BC; then construct a flat triangular mould, having two of its edges perfectly straight and making with each other an angle equal to ABC. Each of the edges should be a little longer than the chord AC. In the points A, C fix pins; and fix a pencil to the mould at B, and move the mould so as to keep its edges touching the pins at A and C, when the pencil will describe the required arc.

7. *Another method.* (Fig. 18.)

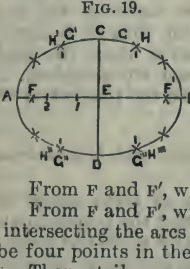
FIG. 18.



Draw the chord ADC, and draw EBF parallel to it; bisect the chord in D and draw DB perpendicular to AC; join AB and BC; draw AE perpendicular to AB and CF perpendicular to BC; also draw A*n* and C*n* perpendicular to AC; divide AC and EF into the same number of equal parts, and A*n*, C*n* into half that number of equal parts; join 1 and 1, 2 and 2, also B and *s*, *s*, and B, and *t*, *t*; through the points where they intersect describe a curve, which will be the arc required.

8. *To describe an ellipse, the major and minor axes being given.* (Fig. 19.)

FIG. 19.



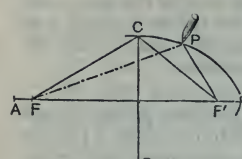
Let AB be the major and CD the minor axis, bisecting each other at right angles in the centre E; from C as a centre, with EA as radius, describe arcs cutting AB in F and F′, which will be the foci of the ellipse; between E and F set off any number of points, as 1, 2 (it is advisable that these points should be closer as they approach F).

From F and F′, with radius B1, describe the arcs G, G′, G″, G‴.

From F and F′, with radius A1, describe the arcs H, H′, H″, H‴, intersecting the arcs G, G′, G″, G‴ in the points I, I, I, I, which will be four points in the curve.

Then strike arcs from F, F′ first with A2, then with B2; these radii intersecting will give four more points. Proceed in this way with all the points between E and F; the curve of the ellipse must then be traced through these points by hand.

FIG. 20.



9. *Another method.* (Fig. 20.)

Let AB and CD be the axes; find F, F′, the two foci as before; join CF, CF′; make an endless thread equal in length to the perimeter of the triangle CFF′, and passing it round two drawing-pins at F and F′, draw it taut by means of a pencil-point P, so as to make a triangle PFF′ equal in perimeter to CFF′; move the pencil-point P along, keeping the thread taut, and the required curve will be described.

FIG. 21.

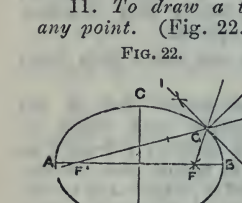
10. *Another method.* (Fig. 21.) (Approximate.)

At *o*, the intersection of the two diameters, as a centre, with a radius equal to the difference of the semi-diameters, describe the arc *ab*, and from *b* as a centre with half the chord *bca* describe the arc *cd*; from *o* as centre with the distance *od* cut the diameters in *dr*, *dt*; draw the lines *rs*, *rs*, *ts*, *ts*, then from *r* and *t* describe the arcs SDS, SCS; also from *d* and *d* describe the smaller arcs SAS, SBS, which will complete the ellipse required.

This method is applicable when the minor axis is at least $\frac{2}{3}$ the major.

11. *To draw a tangent and a normal to an ellipse at any point.* (Fig. 22.)

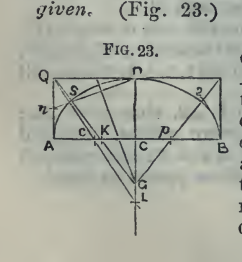
FIG. 22.



Let G be the point; from F, F′, the two foci of the ellipse, draw straight lines through G and produce them; bisect the angle made by the produced parts, by GH, then GH is normal to the curve; at G bisect the angle formed by FG and F′G produced, by IJ, then IJ will be the tangent to the curve at G.

12. *To describe an elliptic arc, the span and height being given.* (Fig. 23.) (Approximate.)

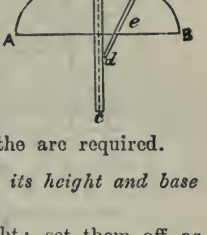
FIG. 23.



Bisect with a line at right angles the chord or span AB; erect the perpendicular AQ, and draw the line QD equal and parallel to AC; bisect AC in *c*, and AQ in *n*; make CL equal to CD, and draw the line LcQ; draw also the line *n*SD, and bisect SD with a line KG at right angles to it, and meeting the line LD in G; draw the line GKQ, and make C*p* equal to CK, and draw the line G*p*2; then from G as centre with the radius

GD describe the arc SD2, and from K and p as centres with the radius AK describe the arcs AS and 2B, which complete the arc, as required. This method gives good results for ellipses of all proportions.

13. *Another method.* (Fig. 24.)

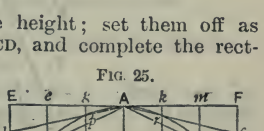


Bisect the major axis AB, and fix at right angles to it a straight guide, as *bc* ; prepare of any material a rod or staff, *def* ; at *f* fix a pencil or tracer so that *df* is one-half AB, and at E fix a pin so that *ef* is one-half the minor axis ; move the staff, keeping its end *d* to the guide, and the pin *e* to AB, and the tracer will describe a half of the arc required.

14. *To describe a parabolic arc when its height and base are given.* (Fig. 25.)

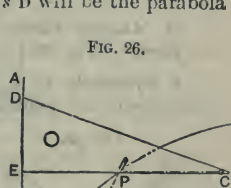
Let CD be the base and AB the height; set them off as shown in the figure, so that CB = CD, and complete the rectangle CDFE; divide EC and FD into any number of equal parts, say three, at *a*, *b*, *c*, and *d*; join A*a*, A*b*, A*c*, A*d*; divide AE, AF, BC, and BD into the same number of equal parts at *e*, *g*, *k*, *m*, *f*, *h*, *l*, *n*; join *ef*, *gh*, *kl*, *mn*, cutting A*b*, A*a*, A*c*, A*d* at *q*, *p*, *r*, and *s*. A line drawn through C *q p* A *r s* D will be the parabola required.

FIG. 25.



15. *Another method, when the directrix and focus are given.* (Fig. 26.)

FIG. 26.



Place a straight-edge to the directrix AB, and apply to it a square CDE; to the end C of the square fasten a thread, and pin the other end to S the focus, making the length of the thread equal to CE; slide the square along the straight-edge, holding the thread taut against the edge of the square by a pencil P, by which the curve is described.

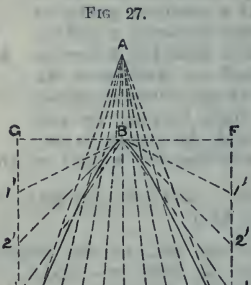
16. *To describe a hyperbola, the diameter, abscissa, and double ordinate being given.* (Fig. 27.)

Let AB be the diameter, BC its abscissa, and DE its double ordinate; then through B draw GF parallel and equal to DE; draw also DG and EF parallel to the abscissa BC.

Divide DC and CE into the same number of equal parts, as 1,2, &c., and from the points of division draw lines meeting in A.

Divide GD and EF each into the same number of parts as DC or CE, and from the points of division 1′, 2′, &c., draw lines meeting in B.

FIG. 27.

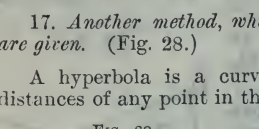


The points of intersection of the lines 1 and 1′, 2 and 2′, &c., thus found, will be points in the required curve.

17. *Another method, when the foci and a point on the curve are given.* (Fig. 28.)

A hyperbola is a curve such that the difference of the distances of any point in the curve from the two foci is equal to the transverse axis; and this property suggests the following mechanical construction:—

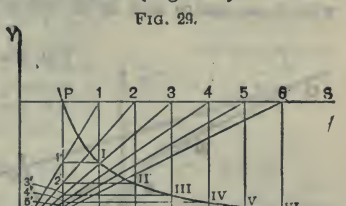
FIG. 28.



Let P (fig. 28) be any point on the curve, and F and F_1 its foci; join FF_1 and produce it, making XX′ the axis; draw PM perpendicular to XX′, and produce it to Q, making MQ equal to PM; bisect FF_1 at C, and produce PC, QC, to CP′, CQ′, making them equal to CP or CQ. The P, P′, Q, Q′ are four points on the curve. From one of them, say P, stretch two pieces of string PF and PF_1, fastening them to the paper at F and F_1, and simply knotting them at P; slip a small bead over them at P, and taking hold of P and keeping the thread taut, slide the bead along the threads, and the bead will describe the curve as far as the axis. Repeat this process at P′, Q, and Q′.

18. *To describe a rectangular hyperbola, given the asymptotes and a point on the curve.* (Fig. 29.)

FIG. 29.

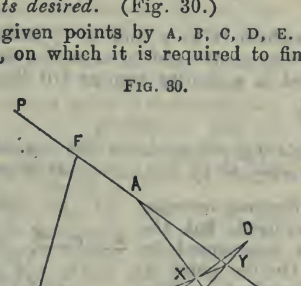


Let OX, OY be the asymptotes, and P the given point. Draw PM parallel to OY, and PS parallel to OX; set off any ordinates (generally equidistant for convenience) 11, 22, 33, 44, 55, 66, and join O to the intersections of these ordinates with PS, cutting PM at 1′, 2′, 3′, etc.; through 1′ draw 1′I parallel to OX, cutting 11 in I; through 2′, 2′II cutting 22 in II, and so on for III, IV, V, and VI; then P, I, II, III, etc., are points on the required curve.

19. *Given five points on any conic to obtain any number of additional points desired.* (Fig. 30.)

Denote the given points by A, B, C, D, E. Draw any line PAP′ through A, on which it is required to find a sixth point.

FIG. 30.



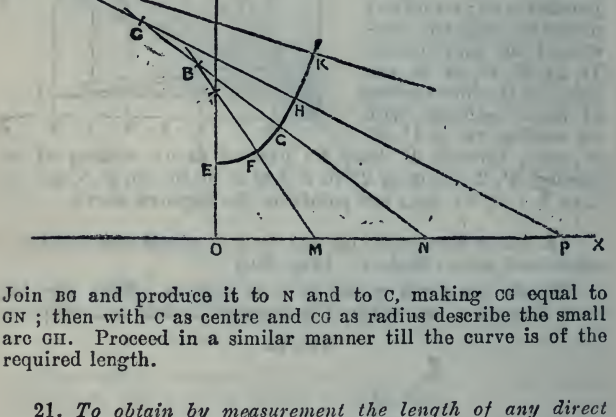
Join AB, DE, cutting at X, and CD cutting PAP′ at Y. Join BC, cutting XY in Z; join EZ, cutting PAP′ in F. F is the required point on the curve; by drawing additional lines through A, any number of points on the curve may be obtained.

20. *To construct a catenary approximately.* (Fig. 30A.)

Let E be the lowest point in the curve, OE its parameter, and OX its directrix. Make AE equal to OE; then with A as

centre and AE as radius describe the small arc EF. Join FA and produce it to M and to B, making BF equal to FM; then with B as centre and BF as radius describe the small arc FG.

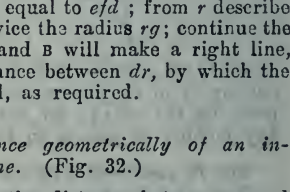
FIG. 30A.



Join BG and produce it to N and to C, making CG equal to GN ; then with C as centre and CG as radius describe the small arc GH. Proceed in a similar manner till the curve is of the required length.

21. *To obtain by measurement the length of any direct line, though intercepted by some material object.* (Fig. 31.)

FIG. 31.



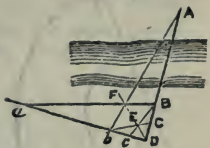
Suppose the distance between A and B is required, but the straight line is intercepted by the object G. On the point *d*, with any convenient radius, describe the arc *cc'*, and make the arc twice the radius *dc* in length ; through *c'* draw the *line dc'e*, and on *e* describe another arc *ff* equal in length to the radius *dc* ; draw the line *efr* equal to *efd* ; from *r* describe the arc *g'g*, equal in length to twice the radius *rg*; continue the line through *rg* to B : then A and B will make a right line, and *de* or *er* will equal the distance between *dr*, by which the distance between AB is obtained, as required.

22. *To ascertain the distance geometrically of an inaccessible object on a level plane.* (Fig. 32.)

Let it be required to find the distance between A and B, A being inaccessible. Produce AB to any point D, and

bisect BD in C; through D draw D*a*, making any angle with DA, and take DC and DB respectively and set them off on D*a* as D*b* and D*c*; join B*c*, C*b*, and A*b*; through E, the intersection of B*c* and C*b*, draw DEF meeting A*b* in F; join BF and produce it till it meets D*a* in *a*: then *ab* will be equal to AB, the distance required.

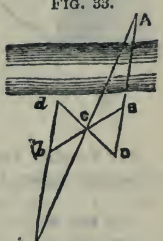
FIG. 32.



23. *Another method.* (Fig. 33.)

Produce AB to any point D; draw the line D*d* at any angle to the line AB; bisect the line D*d* in C, through which draw the line B*b*, and make C*b* equal to BC; join AC and *db*, and produce them till they meet at *a*: then *ba* will equal BA, the distance required.

FIG. 33.

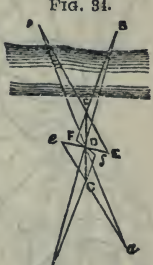


24. *To measure the distance between two objects, both being inaccessible.* (Fig. 34.)

Let it be required to find the distance between the points A and B, both being inaccessible. From any point C draw any line C*c*, and bisect it in D; produce A*c* and B*c*, and prolong them to E and F; take the point E in the prolongation of A*c*, and draw the line ED*e*, making D*e* equal to DE.

In like manner take the point F in the prolongation of B*c*, and make D*f* equal to DF; produce AD and *e*C till they meet in *a*, and also produce BD and *f*C till they meet in *b*: then the distance between the points *a* and *b* equals the distance between the inaccessible points A and B.

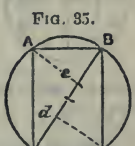
FIG. 34.



25. *To cut a beam of the strongest section from any round piece of timber.* (Fig. 35.)

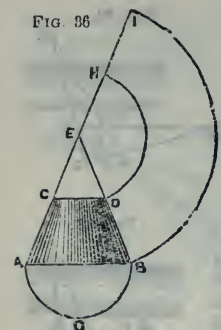
Divide any diameter CB of the circle into three equal parts; from *d* or *e*, the two points of division in CB, erect a perpendicular cutting the circumference of the circle in D or A; draw CD and DB, also AC equal to DB and AB equal to CD: the rectangle ABCD will be the section of the beam required.

FIG. 35.



Note.—To get the *stiffest* beam make $Cd = \frac{1}{4}$ CB and proceed as before.

FIG. 36

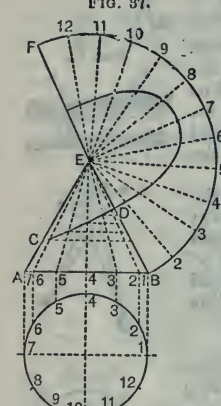


26. *To describe the proper form of a flat plate by which to construct any given frustum of a cone.* (Fig. 36.)

Let ABCD represent the required frustum of a cone ; continue the lines AC and BD till they meet in E ; from E as a centre, with ED as radius, describe the arc DH, and from the same centre, with EB as radius, describe the arc BI; make BI equal in length to twice AGB, equal to the circumference of the base of the cone ; draw the line EI : then BDHI is the form of the plate required.

27. *To find the development of the frustum of a right cone when cut by an angle inclined to the base.* (Fig. 37.)

FIG. 37.



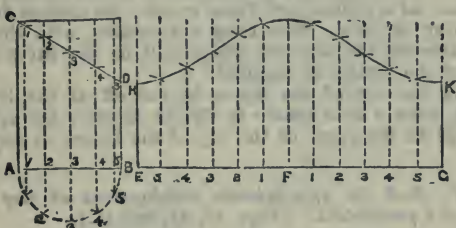
Let ABCD represent the required frustum of the cone ; continue the lines AC and BD till they meet at E ; divide the circumference of the base of the cone into any number of equal parts—say 12—in the points 1, 2, 3, etc. ; join the projections of these points to E ; next find the development of the base of the cone, as shown in the preceding example, and on it set off the same number of points—viz. 12—and draw lines from them to E ; project the points of intersection of each of the lines E1, E2, E3, etc., with the line CD, horizontally on to either of the slant sides (say EB) ; then from E as centre measure the distance down along EB to the foot of each projection and set it off on the corresponding numbers (measuring from E) in the development : a line drawn through these points will give the curve of the top of the section, as required.

28. *To find the development of the frustum of a cylinder when cut by a plane inclined to the base.* (Fig. 38.)

Let ABCD represent the required frustum of a cylinder ; divide the base into any number of equal parts—say 12—

and draw lines through those points on the cylinder parallel to AC and BD ; draw a line EFG equal in length to the circumference of the cylinder, and divide it into the same number of parts ; on each point of division set up perpendiculars to

FIG. 38.



it, making EH and GK equal in length to BD, and make FI equal in length to AC; then take the height at 1 and set it up on the corresponding number on each side of FI, and so on with each number : a line traced through the points thus obtained will be the curve of the required development.

29. *To find the approximate development of any given portion of a segment of a sphere.* (Figs. 39, 40, and 41.)

FIG. 39.

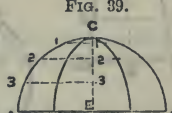


FIG. 41.

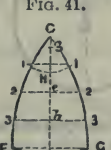
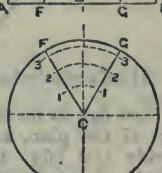


FIG. 40.



Let ABC (fig. 39) be the middle section of the segment, and CFG in the plan (fig. 40) the portion to be developed; bisect AB (fig. 39) in E, and set up the perpendicular EC ; divide the arc AC into any given number of equal parts—say, four—and through the points of division draw the lines 1 1, 2 2, etc., parallel to AB ; on the plan (fig. 40) from C as a centre, with the radius 1 1 taken from fig. 39, draw the arcs 1 1 cutting FC and CG in 1 and 1, and so on with 2 2 and 3 3; draw any line BC (fig. 41), making it equal in length to BC (fig. 39), and on it set off the same number of equal parts; at each point of division draw lines perpendicular to BC, and number them the same as on fig. 39.

Measure the length of the arc 1 1 in fig. 40, and set off half of it on each side of BC on line 1 1, and so on with each arc, including FG ; a line traced through the points thus obtained will give the curve of the sides of the given portion of the segment when it is developed. To describe the curve at the bottom of the figure, take one-fourth of the circumference of the base as a radius, and from F and G as centres describe arcs cutting BC in S ; then from S as centre, with the same radius, describe the arc FBG, which will be the curve of the bottom of the figure, as required.

Should the top of the figure be cut off at the line 1 1 (fig. 39), from S as a centre in fig. 41 describe the arc 1H1, which will be the curve of the top of the figure, as required.

30. *To find the approximate development of any given portion of a paraboloid.* (Figs. 42, 43, and 44.)

FIG. 41.

FIG. 42.

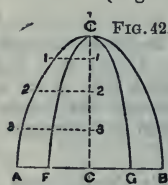


FIG. 43.

The development is found in the same manner as that of a portion of a segment of a sphere, as described in the last example (No. 29), with but one exception—that is, the length of the radius for describing the bottom curve of the figure, which instead of being equal to one-fourth of the circumference, as in example No. 29, is equal to one-half the length of the arc ACB (fig. 42) in this example.

31. *To find the development of an entablature plate.*

Let fig. 45 be the side elevation, fig. 46 the front elevation, fig. 47 the plan, and fig. 48 the development of the figure ; divide ADC (fig. 46) into eight equal parts, and from the points of intersection draw lines parallel to ABC, cutting CD (fig. 45) in the points 1, 2, etc.; on BD (fig. 45) erect a perpendicular EC, and from the points on CD draw lines parallel to BED. From fig. 46 take the points 1, 2, etc., on ABC and set them off on AFC (fig. 47), and erect perpendiculars from AFC at these points. From C (fig. 45) along CE measure the points C, 1, C, 2, etc., and set them off on their corresponding lines from AFC in fig. 47; draw a line through those points, then measure it with its divisions and set it off in fig. 48 as a straight line

AEC, and at the points of division erect perpendiculars, continuing them either side of the line AEC ; measure the distances 1, 1; 2, 2, etc. (fig. 45), on either side of CE, and set them off

FIG. 45.

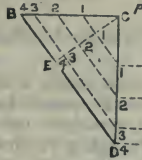


FIG. 46.

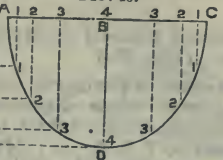


FIG. 47.

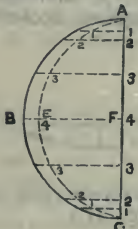
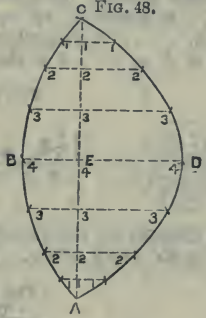


FIG. 48.



from AEC (fig. 48) on their corresponding lines, and on their respective sides of AEC. These will give the development.

32. *To describe a cycloid, the generating circle being given.* (Fig. 49.)

Let B6 be the generating circle ; draw a line ABC, equal to the circumference of the generating circle, by dividing the circle into any number of given parts, as 1, 2, 3, etc., and

FIG. 49.

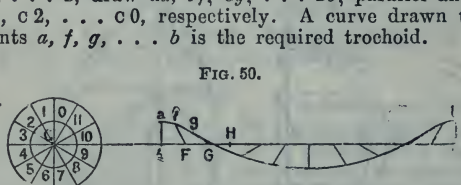
setting off half that number of parts on each side of B; draw lines from the intersections of the circle 1, 2, 3, etc., 7, 8, 9,

etc., parallel to AC ; set off one division of the circle outwards on the first lines 5 and 7, two divisions on the next lines 4 and 8, then three on the next, and so on : then the intersection of those points on the lines 1, 2, 3, etc., will be points in the curve.

33. *To draw a trochoid or wave-form, the height and length being given.* (Fig. 50.)

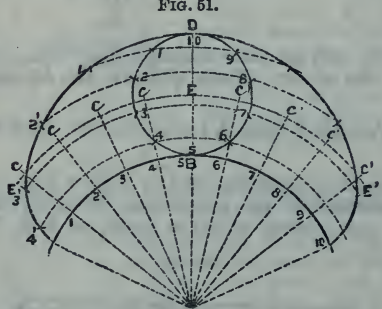
Draw AB equal to the length ; with centre C on AB produced describe a circle whose diameter is equal to the height. Divide the circumference into a convenient number (say 12) of equal parts 0, 1, 2, 3, . . . , C0 being vertical. Divide AB into the same number of equal parts F, G, H, . . . From A, F, G, . . . B, draw A*a*, F*f*, G*g*, . . . B*b*, parallel and equal C 0, C 1, C 2, . . . C 0, respectively. A curve drawn through the points *a, f, g, . . . b* is the required trochoid.

FIG. 50.



34. *To describe an epicycloid, the generating circle and the directing circle being given.* (Fig. 51.)

FIG. 51.



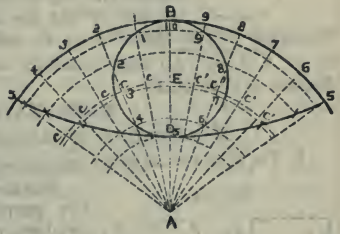
Let BD be the generating circle, and AB the directing circle; divide the generating circle into any number of equal parts (say 10) as 1, 2, 3, etc., and set off the same distances round the directing circle; draw radial lines from A through these last points, and produce them to an arc drawn with A as centre and AE as radius, as shown by *cccc* and *c'c'c'c'* on the diagram ; draw

concentric arcs also through all the points on the generating circle, with A as centre; then taking c, c, c, c and c', c', c', c' as centres, and BE as radius, describe arcs cutting the concentric circles at 1′, 2′, etc.: the points thus found will be points in the required curve.

35. *To describe a hypocycloid, the generating circle and the directing circle being given.* (Fig. 52.)

Proceed as in the epicycloid, the exception being that the construction lines are drawn within the directing circle instead of outside, as in the epicycloid.

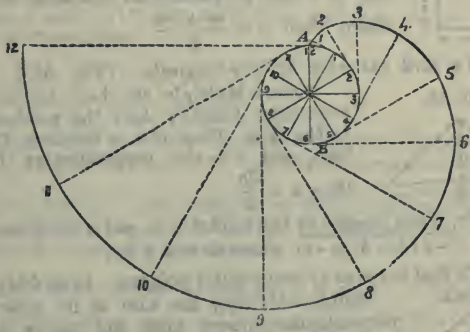
FIG. 52.



36. *To describe the involute of a circle.* (Fig. 53.)

Let AB be the given circle, which divide into any equal number of parts (say 12) as 1, 2, 3, etc.; from the centre draw radii to these points; then draw lines (tangents) at right angles to these radii. On the tangent to radius No. 1 set off from the circle a distance equal to one part, and on each of the

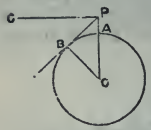
FIG. 53.



tangents set off the number of parts corresponding to the number of its radius, so that No. 12 has twelve divisions set off on it (that is, equal to the circumference of the circle): a line traced through the ends of these lines will be the curve required.

37. *To find the dip of the horizon.* (Fig. 53A.)

FIG. 53A.



Let O denote the centre of the earth, PB a tangent from the eye of an observer looking from a height AP to the earth's surface at B; then B is a point on the horizon: draw PC at right angles to PO; then the angle BPC is called the dip of the horizon.

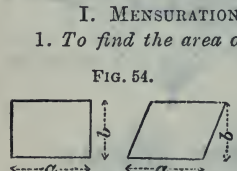
Let OP cut the earth's surface at A, and let the angle BPC be denoted by θ; with distances in miles, PB $= \sqrt{2 \cdot AP \cdot AO}$ approximately $= \sqrt{8{,}000 \times AP}$; and θ in degrees $= 1{\cdot}28\sqrt{AP}$.

MENSURATION.

I. MENSURATION OF AREAS AND PERIMETERS.

1. *To find the area of any parallelogram.* (Fig. 54.)

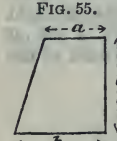
FIG. 54.



RULE.—Multiply the length by the perpendicular height, and the product will be the area. Thus, if A = the area, a = the length, and b = the perpendicular height, then $A = ab$.

2. *To find the area of a trapezoid.* (Fig. 55.)

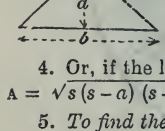
FIG. 55.



RULE.—Multiply the sum of the parallel sides by the perpendicular distance between them; half the product will be the area. Thus, if A = the area, b and a = the parallel sides, and c = the perpendicular distance between them, then $A = \dfrac{(a+b)c}{2}$

3. *To find the area of any triangle.* (Fig. 56.)

FIG. 56.

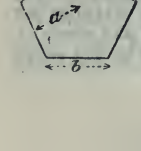


RULE.—Multiply the base by the perpendicular height; half the product will be the area. Thus, if A = the area, b = the base, and d = the perpendicular height, then $A = \dfrac{db}{2}$

4. Or, if the lengths of the 3 sides a, b, and c are given, then $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $2s = a + b + c$.

5. *To find the area of any regular polygon.* (Fig. 56A.)

FIG. 56A.



RULE.—Multiply the sum of its sides by a perpendicular drawn from the centre of the polygon to one of its sides; half the product will be the area. Thus if A = the area, c = the number of sides, b = the length of one side, and a = the perpendicular, then $A = \dfrac{abc}{2}$

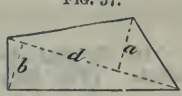
Table of Regular Polygons.

A = the angle contained between any two sides.
R = the radius of the circumscribed circle.
r = the radius of the inscribed circle.
S = the side of the polygon.

No. of Sides	Name	A	R = S ×	r = S ×	S = R ×	S = r ×	Area = S^2 ×
3	Trigon	60°	·57735	·28868	1·73205	3·46410	·43301
4	Tetragon	90°	·70711	·50000	1·41421	2·00000	1·00000
5	Pentagon	108°	·85065	·68819	1·17557	1·45309	1·72048
6	Hexagon	120°	1·00000	·86603	1·00000	1·15470	2·59808
7	Heptagon	$128\frac{4}{7}°$	1·15238	1·03826	·86777	·96315	3·63391
8	Octagon	135°	1·30656	1·20711	·76537	·82843	4·82843
9	Nonagon	140°	1·46190	1·37374	·68404	·72794	6·18182
10	Decagon	144°	1·61803	1·53884	·61803	·64984	7·69421
11	Undecagon	$147\frac{3}{11}°$	1·77473	1·70284	·56347	·58725	9·36564
12	Duodecagon	150°	1·93185	1·86603	·51764	·53590	11·19615

6. *To find the area of a quadrilateral.* (Fig. 57.)

Fig. 57.



Rule.—Multiply the diagonal d by the sum of the two perpendiculars a and b let fall upon it from the opposite angles; half the product will be the area. Thus if A = the area, a and b = the perpendiculars, and d = the diagonal, then

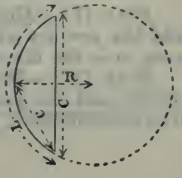
$$A = \frac{(a+b)\,d}{2}.$$

7. *To find the circumference of a circle, the diameter being given; or to find the diameter of a circle, the circumference being given.*

Rule.—Multiply the diameter by 3·1416, the product will be the circumference; or divide the circumference by 3·1416, the quotient will be the diameter.

8. *To find the length of any arc of a circle.* (Fig. 58.)

Fig. 58.



Rule (1).—From eight times the chord of half the arc subtract the chord of the whole arc; one-third of the remainder will be the length of the arc, nearly. Thus if L = length of the arc, C = chord of the whole arc, c = chord of half the arc, then

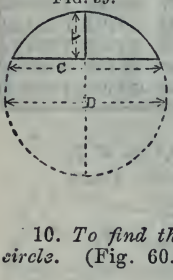
$$L = \frac{8c - C}{3}$$

RULE (II).—The radius being known, multiply together the number of degrees in the arc, the radius, and the number ·01745 ; the product will be the length of the arc. Thus if L = length of the arc, D = degrees in the arc, R = radius, then

$$L = D \times R \times \cdot 01745.$$

RULE (III).—(*Applicable to any fairly flat curve.*) Add to the chord eight-thirds the square of the maximum height (or versed sine) divided by the chord. The sum is the length of the curve, very nearly. Thus if C = chord, and V = greatest height of arc above chord, length $= C + \frac{8}{3}\frac{V^2}{C}$

FIG. 59.



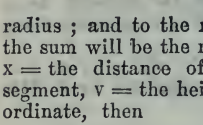
9. *To find the diameter of a circle, the chord and versed sine being given.* (Fig. 59.)

RULE.—Divide the square of half the chord by the versed sine, to the quotient add the versed sine, and the sum will be the diameter. Thus if D = the diameter, C = the chord, and V = the versed sine, then

$$D = \left\{\frac{\left(\frac{C}{2}\right)^2}{V} + V\right\}$$

10. *To find the length of any ordinate of a segment of a circle.* (Fig. 60.)

FIG. 60.



RULE.—Find the radius of the arc of the segment (if not given) by the preceding formula; and from the square root of the difference of the squares of the radius and distance of the ordinate from the centre of the segment, subtract the radius ; and to the result add the height of the segment, and the sum will be the required ordinate. Thus if R = the radius, X = the distance of the ordinate from the centre of the segment, V = the height of the segment, and Y = the required ordinate, then

$$Y = \sqrt{R^2 - X^2} - R + V.$$

11. *To find the area of a circle.*

RULE (I).—Multiply the square of the diameter by ·7854, and the product will equal the area, nearly. Thus if A = the area, D = the diameter, then $A = D^2 \times \cdot 7854$.

RULE (II).—Multiply the square of the circumference by ·07958, and the product will be the area. Thus if A = area, C = circumference, then $A = C^2 \times \cdot 07958$.

Table of Properties of the Circle.

$\pi = 3{\cdot}14159265358979323846$	$\sqrt{2} = 1{\cdot}41421356237309504880$
$\frac{\pi}{2} = 1{\cdot}57079632679489661923$	$\sqrt{\frac{1}{2}} = {\cdot}70710678118654752440$
$\frac{\pi}{4} = {\cdot}78539816339744830962$	$2\sqrt{\pi} = 3{\cdot}54490770181103205460$
$\frac{\pi}{6} = {\cdot}52359877559829887308$	$2\sqrt{\frac{1}{\pi}} = 1{\cdot}12837916709551257390$
$\pi\sqrt{2} = 4{\cdot}44288293815836624702$	$\frac{1}{2}\sqrt{\pi} = {\cdot}88622692545275801365$
$\pi\sqrt{\frac{1}{2}} = 2{\cdot}22144146907918312351$	$\frac{1}{8}\sqrt{\frac{1}{\pi}} = {\cdot}07052369794346953587$
$\sqrt{\pi} = 1{\cdot}77245385090551602730$	$2\pi = 6{\cdot}28318530717958647693$
$\sqrt{\frac{1}{\pi}} = {\cdot}56418958354775628695$	$\frac{2}{\pi} = {\cdot}63661977236758134308$

$$\frac{\pi}{180} = {\cdot}01745 \qquad \frac{180}{\pi} = 57{\cdot}3$$

$$\frac{1}{\pi} = {\cdot}3183 \qquad \frac{60}{2\pi} = 9{\cdot}5493$$

$$\pi^2 = 9{\cdot}870$$

In the following formulæ A = area, C = circumference, D = diameter, S = side of square.

Circumference $= D \times \pi = R \times 2\pi = \sqrt{A} \times 2\sqrt{\pi}$

Diameter $= C \times \frac{1}{\pi} = \sqrt{A} \times 2\sqrt{\frac{1}{\pi}}$

Radius $= C \times \frac{1}{2\pi} = \sqrt{A} \times \sqrt{\frac{1}{\pi}}$

Area $= R^2 \times \pi = D^2 \times \frac{\pi}{4} = \frac{1}{2}\,RC$

Side of equal square $= R \times \sqrt{\pi} = D \times \frac{1}{2}\sqrt{\pi} = C \times \frac{1}{2}\sqrt{\frac{1}{\pi}}$

Side of inscribed square $= D \times \sqrt{\frac{1}{2}} = C \times \frac{1}{\pi}\sqrt{\frac{1}{2}} = \sqrt{A} \times \sqrt{\frac{2}{\pi}}$

Diameter of equal circle $= S \times 2\sqrt{\frac{1}{\pi}}$

Diameter of circumscribing circle $= S \times \sqrt{2}$

Circumference of circumscribing circle $= S \times \pi\sqrt{2}$

Circumference of equal circle $= S \times 2\sqrt{\pi}$

Area of inscribed square $= A \times \frac{2}{\pi}$

12. *To find the area of a sector of a circle.*

RULE (I.)—Multiply the length of the arc by the radius of the sector, and half the product will equal the area.

A = area of sector, L = length of arc, R = radius.

RULE (II).—Multiply the number of degrees in the arc by the area of the circle, and $\frac{1}{360}$ of the product will equal the area. Thus, if A = area, D = number of degrees in the arc a = area of circle, then

$$A = \frac{Da}{360}$$

13. *To find the area of the segment of a circle.*

RULE (I.)—Find the area of a sector having the same arc as the segment; then deduct the area of the triangle contained between the chord of the segment and the radii of the sector. The remainder will be the area of the segment. Thus, if A = the area of the segment, c = the chord, and H = the height, then

$$A = \frac{1}{2H^2}\left[\left\{\left(\frac{c}{2}\right)^2 + H^2\right\}^2 \tan^{-1}\frac{2H}{c} - \frac{cH}{2}\left\{\left(\frac{c}{2}\right)^2 - H^2\right\}\right]$$

RULE (II).—To two-thirds of the product of the chord and height of the segment, add the cube of the height divided by twice the chord; the sum will be the area of the segment, nearly. Thus,

$$A = \left(\frac{2cH}{3} + \frac{H^3}{2c}\right)$$

FIG. 61.

14. *To find the area of a circular zone.* (Fig. 61.)

RULE.—Find the area of the circle of which the zone forms a part, and from it subtract the sum of the two segments of the circle formed by the zone; the remainder will be the area. Thus, if A = area of the zone, a and b = the area of the two segments respectively, and C = area of the circle, then $A = C - (a + b)$.

15. *To find the area of a flat circular ring.* (Fig. 62.)

RULE.—Multiply the sum of the inside and outside diameters by their difference, and the result by ·7854;

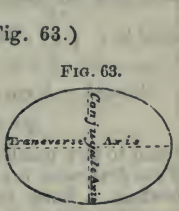
FIG. 62.

the product last obtained will be the area. Thus, if A = area of ring, D = diameter of large circle, and d = diameter of small circle, then

$$A = \cdot 7854\{(D + d)\ (D - d)\}$$

16. *To find the area of an ellipse.* (Fig. 63.)

FIG. 63.



RULE.—Multiply together the transverse and conjugate diameters of the ellipse, and the result by $\frac{\pi}{4}$ or ·7854; the product will be the area. Thus, if A = area of ellipse, a = the conjugate diameter, and b = the transverse diameter, then

$$A = \cdot ab \times \cdot 7854.$$

17. *To find the area bounded by a rectangular hyperbola, two ordinates and the base.* (Fig. 8, p. 15.)

RULE.—Multiply the product of either ordinate and the corresponding abscissa by the hyperbolic logarithm of the ratio between the two abscissæ. Thus the area of ABCD is equal to AB × OB $\times \log_e \frac{OC}{OB}$

18. *To find the area bounded by a cycloid and the line joining the cusps.* (Fig. 11, p. 18.)

RULE.—Multiply the area of its generating circle by 3; *or* multiply the product of its length and height by $\frac{3}{4}$.

19. *To find the area bounded by a trochoid and a line joining the crests.* (Fig. 50, p. 34.)

RULE.—If R be the radius of the rolling circle (or the length divided by 2π), and r the radius of the tracing circle (or one-half the height from crest to trough), the required area is equal to $\pi r\ (2R + r)$.

Note.—The area between the curve and the line joining the troughs is $\pi r\ (2R - r)$.

20. *To find the area of a segment of a parabola.*

RULE.—Multiply the base by $\frac{2}{3}$ of the maximum height.

21. *To find a general expression for the area of any plane curve.*

Using cartesian co-ordinates, the area intercepted between the curve, the x axis, and two ordinates distant a and b from the origin, is equal to the definite integral $\int_a^b y \,.\, dx$.

Using polar co-ordinates, the area intercepted between the curve and two radial lines making angles α and β with Ox, is equal to $\frac{1}{2}\int_\alpha^\beta r^2 \,.\, d\theta$.

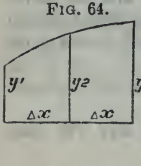
Remark.—A curve whose equation is given by

$$y = a + bx + cx^2 + dx^3 + \ldots Kx^n$$

is said to be a parabolic curve of the n^{th} order. Thus a parabolic of the first order is a straight line; of the second order a common parabola. Rules for the area of a parabola of any order are applicable also to curves of a lower order, but not in general to those of a higher order.

22. *To find the area of a parabola of the third order when three ordinates are given.* (Fig. 64.)

FIG. 64.



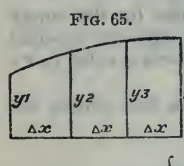
RULE.—To the sum of the two endmost ordinates add four times the intermediate ordinate; multiply the final sum by $\frac{1}{3}$ of the common interval between the ordinates. The result will be the area. Thus, if y_1, y_2, and y_3 be the ordinates, Δx the common interval, and $\int y dx$ the area, then

$$\int y dx = \frac{\Delta x}{2}(y_1 + 4y_2 + y_3).$$

Note.—This is termed Simpson's first rule.

23. *To find the area of a parabola of the third order when four ordinates are given.*

FIG. 65.



RULE.—To the sum of the two endmost ordinates add three times the intermediate ordinates; multiply the final sum by $\frac{3}{8}$ of the common interval between the ordinates: the result will be the area. Thus, if $\int y dx$ = the area, then

$$\int y dx = \frac{3\Delta x}{8}(y_1 + 3y_2 + 3y_3 + y_4).$$

Note.—This is termed Simpson's second rule.

TABLE SHOWING THE MULTIPLIERS FOR THE FOREGOING AND SOME OTHER RULES.

y_1, y_2, y_3, etc. = the ordinates, and Δx = the common interval or abscissa between the ordinates.

1. Trapezoidal rule.

$$\text{Area} = \frac{\Delta x}{2}(y_1 + y_2).$$

2. Rule for parabola of the third order with three ordinates.

$$\text{Area} = \frac{\Delta x}{3}(y_1 + 4y_2 + y_3).$$ (Simpson's first rule.)

3. Rule for parabola of the third order with four ordinates.

$$\text{Area} = \frac{3\Delta x}{8}(y_1 + 3y_2 + 3y_3 + y_4).$$ (Simpson's second rule.)

4. Rule for parabola of the fifth order with five ordinates.

$$\text{Area} = \frac{2\Delta x}{45}(7y_1 + 32y_2 + 12y_3 + 32y_4 + 7y_5).$$

5. Rule for parabola of the fifth order with six ordinates.

$$\text{Area} = \frac{5\Delta x}{288}(19y_1 + 75y_2 + 50y_3 + 50y_4 + 75y_5 + 19y_6).$$

6. Rule for parabola of the seventh order with seven ordinates.

$$\text{Area} = \frac{\Delta x}{140}(41y_1 + 216y_2 + 27y_3 + 272y_4 + 27y_5 + 216y_6 + 41y_7).$$

7. Approximation for curve with six ordinates.

$$\text{Area} = \frac{25\Delta x}{24}(0{\cdot}4y_1 + y_2 + y_3 + y_4 + y_5 + 0{\cdot}4y_6).$$

8. Weddle's approximation for curve with seven ordinates.

$$\text{Area} = \frac{3\Delta x}{10}(y_1 + 5y_2 + y_3 + 6y_4 + y_5 + 6y_6 + y_7).$$

24. *To measure any curvilinear area by means of the trapezoidal rule.*

RULE.—To the sum of half the two endmost ordinates add all the other ordinates, and multiply the sum by the common interval; the result will be the area. Thus,

$$\int y\,dx = \Delta x\left(\frac{y_1 + y_n}{2} + y_2 + y_3 \ldots + y_{n-1}\right)$$

Remark.—In shipbuilding work it is very often convenient to perform the additions in the above rule mechanically, by measuring off the ordinates continuously on a long strip of paper, and measuring the total length on the proper scale. This rule is only approximate, but it is especially useful for getting the areas of the transverse sections in the first rough calculations of trim, displacement, etc.

25. *To measure any curvilinear area by means of Simpson's first rule.*

RULE.—To the sum of the first and last ordinates add four times the intermediate ordinates and twice all the dividing

ordinates; multiply the final sum by $\frac{1}{3}$, the common interval: the result will be the area. Thus

$$\int ydx = \frac{\Delta x}{3}(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 \ldots + 4y_{n-1} + y_n).$$

Remark.—The number of intervals in this rule must be even. The ordinates which separate the parabolas into which the figure is conceived to be divided, are called dividing ordinates, and all the other ordinates except the two endmost ones are called intermediate ordinates.

26. *To measure any curvilinear area by means of Simpson's second rule.*

RULE.—To the sum of the two endmost ordinates add three times the intermediate ordinates and twice all the dividing ordinates; multiply the final sum by $\frac{3}{8}$, the common interval, and the result will be the area. Thus

$$\int ydx = \frac{3\Delta x}{8}(y_1 + 3y_2 + 3y_3 + 2y_4 + 3y_5 \ldots + 3y_{n-1} + y_n).$$

The number of intervals in this case must be a multiple of three.

Remark.—The sequence of the multipliers in the two foregoing rules is obvious. Thus in the first rule the simple multipliers are 1 . 4 . 1, but they are combined thus:—

1 . 4 . 1
1 . 4 . 1
1 . 4 . 1
. . . . . .
1 . 4 . 1
1 . 4 . 1
1 . 4 . 1
1 . 4 . 2 . 4 . 2 . 4 . . . . . 4 . 2 . 4 . 2 . 4 . 1

In the second rule the multipliers are 1 . 3 . 3 . 1.

1 . 3 . 3 . 1
1 . 3 . 3 . 1
1 . 3 . 3 . 1 . . . . .
1 . 3 3 . 1
1 . 3 . 3 . 1
1 . 3 . 3 . 2 . 3 . 3 . 2 . 3 . 3 . . . . . 3 . 3 . 2 . 3 . 3 . 1

And in the same way the multipliers to measure any curvilinear area may be obtained from the table on p. 43.

Simpson's first rule is superior to the second rule in accuracy as well as simplicity.

27. *To measure any curvilinear area when subdivided intervals are used.*

1*st. When Simpson's first rule is used.*

RULE.—Diminish the multiplier of each ordinate belonging to a set of subdivided intervals in the same proportion in which

the intervals are subdivided. Multiply each ordinate by its respective multiplier as thus found, and treat the sum of their products as if they were whole intervals; that is, multiply the sum thus found by $\frac{1}{3}$ of a whole interval, and the product will be the area.

2nd. When Simpson's second rule is used.

RULE.—Proceed as in the first rule, but multiply by $\frac{3}{8}$ of a whole interval for the area.

Example to Simpson's First Rule.—The series of multipliers for whole intervals being 1 . 4 . 2 . 4 . 2, &c., those for half-intervals will be $\frac{1}{2}$, . 2 . 1 . 2 . 1, &c., and for quarter-intervals $\frac{1}{4}$. 1 . $\frac{1}{2}$. 1 . $\frac{1}{2}$, &c.

Remark.—When an ordinate stands between a larger and a smaller interval, its multiplier will be the sum of the two multipliers which it would have had as an end ordinate for each interval. Thus for an ordinate between a whole and a half interval the multiplier is $\frac{1}{2}+1=1\frac{1}{2}$, and between a half and a quarter interval $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$.

TABLE OF MULTIPLIERS WHEN SUBDIVIDED INTERVALS ARE USED.

Simpson's First Rule.

Ordinates	0	1	2	$2\frac{1}{3}$	$2\frac{2}{3}$	3	$3\frac{1}{3}$	$3\frac{2}{3}$	4	5	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8
Multipliers	1	4	$1\frac{1}{3}$	$1\frac{1}{3}$	$\frac{2}{3}$	$1\frac{1}{3}$	$\frac{2}{3}$	$1\frac{1}{3}$	$1\frac{1}{3}$	4	$1\frac{1}{2}$	2	1	2	$\frac{1}{2}$
Ordinates	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4	5	$5\frac{1}{2}$	6	$6\frac{1}{4}$	$6\frac{1}{2}$	$6\frac{3}{4}$	7
Multipliers	$\frac{1}{2}$	2	1	2	1	2	$1\frac{1}{2}$	4	$1\frac{1}{2}$	2	$\frac{3}{4}$	1	$\frac{1}{4}$	1	$\frac{1}{4}$
Ordinates	0	1	2	$2\frac{1}{2}$	3	$3\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$	4	$4\frac{1}{6}$	$4\frac{1}{3}$	$4\frac{1}{2}$	$4\frac{2}{3}$	$4\frac{5}{6}$	5
Multipliers	1	4	$1\frac{1}{2}$	2	$\frac{3}{4}$	1	$\frac{1}{2}$	1	$\frac{5}{12}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$

Simpson's Second Rule.

Ordinates	0	1	2	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6	$6\frac{1}{6}$	$6\frac{1}{3}$	$6\frac{1}{2}$	$6\frac{2}{3}$	$6\frac{5}{6}$	7
Multipliers	1	3	3	$1\frac{1}{2}$	$1\frac{1}{2}$	$1\frac{1}{2}$	1	$1\frac{1}{2}$	$1\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$
Ordinates	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{3}{4}$	3	$3\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$	4
Multipliers	$\frac{1}{3}$	1	1	$\frac{7}{12}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{4}$
Ordinates	0	$\frac{1}{4}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{3}$	$3\frac{2}{3}$	4	$4\frac{1}{6}$	$4\frac{1}{3}$	$4\frac{1}{2}$	$4\frac{2}{3}$	$4\frac{5}{6}$	5
Multipliers	$\frac{1}{2}$	$1\frac{1}{2}$	$1\frac{1}{2}$	1	$1\frac{1}{2}$	$1\frac{1}{2}$	$\frac{5}{6}$	1	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$

Note.—The ordinates in this table are numbered the same as if they were the number of intervals from the origin.

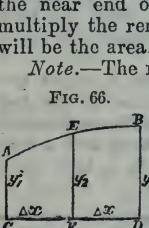
Thomson's Rule may be used when subdivided intervals are used at each end; the advantage being that all multipliers except the three end ones are unity; so also in the common multiplier. Thus the ordinates should be multiplied by $\frac{1}{4}$, $\frac{1}{3}$, $\frac{11}{12}$, 1, 1, 1, ..., 1, 1, $\frac{11}{12}$, $\frac{1}{3}$, $\frac{1}{4}$; the spacing of the three ordinates at each end being one half that elsewhere.

28. *To calculate the area separately of one of the two divisions of a parabolic figure of the second order.* (Fig. 66.)

RULE.—To eight times the middle ordinate add five times the near end ordinate, and subtract the far end ordinate; multiply the remainder by $\frac{1}{12}$ the common interval: the result will be the area.

Note.—The near end ordinate is the ordinate at the end of the division of which the area is to found.

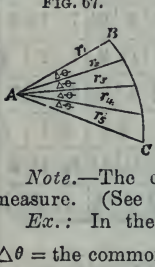
FIG. 66.



Ex.: In figure ABCD let it be required to find the area of the division ACEF. Let y_1 = the near end ordinate, y_2 = the middle ordinate, and y_3 = the far end ordinate; then $\int y dx = \frac{\Delta x}{12}(5y_1 + 8y_2 - y_3)$.

29. *To measure an area bounded by an arc of a plane curve and two radii.* (Fig. 67.)

FIG. 67.



RULE.—Divide the angle subtended by the arc into any number of equal angular intervals by means of radii. Measure these radii and compute their half-squares. Treat those half-squares as if they were ordinates of a curve by Simpson's first or second rule, as the number of intervals may require.

Note.—The common interval must be taken in circular measure. (See pp. 8 and 9.)

Ex.: In the figure ABC let r_1, r_2, r_3, r_4, r_5 = the radii, $\Delta\theta$ = the common angular interval, and $\int \frac{r^2}{2} d\theta$ = the area; then

$$\int \frac{r^2}{2} d\theta = \frac{(r_1^2 + 4r_2^2 + 2r_3^2 + 4r_4^2 + r_5^2)\Delta\theta}{6}$$

30. *To measure any curvilinear area by means of Tchebycheff's rule.*

RULE.—Find the middle point of base, and from it set off, along the base, and in both directions, distances equal to the half length of base multiplied by the constants given in the Schedule below. Erect ordinates at the points so obtained and measure them. Their sum, divided by the number of ordinates, and multiplied by the length of base is the area required.

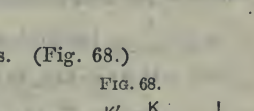
SCHEDULE.	
Number of Ordinates used.	Distance of Ordinates from Middle of Base in Fractions of Half the Base Length.
2	·5773
3	O, ·7071
4	·1876, ·7947
5	O, ·3745, ·8325
6	·2666, ·4225, ·8662
7	O, ·3239, ·5297, ·8839
9	O, ·1679, ·5288, ·6010, ·9116

Note.—As evident from the Schedule, there is an ordinate *at* the middle of base, only when an odd number of ordinates is employed.

Examples.—With *four* ordinates. (Fig. 68.)

Let ABCD be the figure. Bisect the base AB at E. Calling the half length of base b, set off EF and EF′ equal to ·1876 b and EG and EG′ equal to ·7947 b. Erect ordinates GL, FK, F′K′, G′L′ at G, F, F′, G′; and call them y_1, y_2, y_3, and y_4.

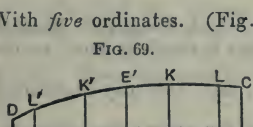
FIG. 68.



Then area of figure ABCD $= \frac{y_1 + y_2 + y_3 + y_4}{4} \times 2b.$

With *five* ordinates. (Fig. 69.)

FIG. 69.



As before, let ABCD be the figure, E the middle of base, and b its half-length. Set off EF and EF′ equal to ·3745 b and EG and EG′ equal to ·8325 b, and erect ordinates at G, F, E, F′, and G′, calling them y_1, y_2, y_3, y_4, y_5.

Then area of figure ABCD $= \frac{y_1 + y_2 + y_3 + y_4 + y_5}{5} \times 2b.$

Note.—This rule can be used for calculating displacements, and fewer ordinates are required for the same degree of accuracy than if Simpson's rule is used. Ten ordinates are usually employed instead of twenty-one, the rule for five ordinates being applied separately to each half of the ship. It is also of great assistance in preparing cross curves of stability. If eight ordinates are used (four repeated), the following "Simpson" sections, assumed numbered from 1 to 21, can be utilized: 2, 5, 7, 10, 12, 15, 17, 20.

31. *To measure any curvilinear area by three ordinates irregularly spaced.*

RULE.—Let ABCDEF (Fig. 70) be the curvilinear area, whose ordinates AB, FC, ED, are y_1, y_2, and y_3. Let AF $= h$, and FE $= hk$, where k is a ratio.

$$\frac{h(k+1)}{6k}\left[y_1 k(2-k) + y_2(k+1)^2 + y_3(2k-1)\right]$$

Note.—If AF $= 2$FE, so that $k = \frac{1}{2}$,

$$\text{Area} = \frac{\text{AE}}{4}(y_1 + 3y_2)$$

32. *To find the area between the first two ordinates of a curvilinear area given three ordinates irregularly spaced.*

RULE.—The area included between the ordinates AB and CF.

$$= \frac{h}{6k(k+1)}\left[y_1 k(3k+2) + y_2(3k+1)(k+1) - y_3\right]$$

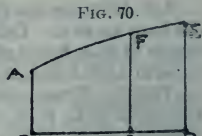
Note.—If AF $= 2$FE, so that $k = \frac{1}{2}$,

$$\text{Area} = \frac{\text{AF}}{18}(7y_1 + 15y_2 - 4y_3)$$

If AF $= \frac{1}{2}$FE, so that $k = 2$,

$$\text{Area} = \frac{\text{AF}}{36}(16y_1 + 21y_2 - y_3)$$

FIG. 70.



33. *To obtain a general expression for the length of any plane curve.*

Using cartesian co-ordinates the length intercepted between two points whose 'x' co-ordinates are a and b is equal to the definite integral $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,.\,dx$. This may be obtained by Simpson's rules in a similar way to the area; the 'ordinate' in this case being replaced by the value of $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ or sec. ϕ, and the common interval being measured along Ox.

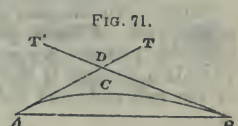
34. *To find approximately the length of any plane curve.* (Fig. 71.)

If the curve is rather irregular, divide it by the eye into any number of fairly flat arcs; join the extremities of each of these arcs by chords. The sum of the length of each of these arcs found by the following rule will be the total length of the curved line.

RULE.—Draw a tangent to the curve at each of its extremities; then take the sum of the two distances from the point of intersection of the two tangents to the extremities of the curve, together with twice the length of the chord ; divide the result by 3 for the length of the arc.

Ex. (fig. 71): Let ACB be one of the arcs, and AB a chord joining the two extremities, and AT, BT′ tangents to the curve at its extremities, cutting each other in D ; then the length of the curve

FIG. 71.



$$ACB = \tfrac{1}{3}(AD + DB + 2AB).$$

Alternatively, for a flat curve see Rule III, p. 38.

35. *To find the perimeter of an ellipse of moderate eccentricity.*

RULE.—If $2a$ is the major and $2b$ the minor axis, where $\frac{b}{a}$ is not very small, then the perimeter is equal to $\frac{\pi}{2a}(3a^2 + b^2)$ very nearly.

36. *To find the length of the evolute of a curve.*

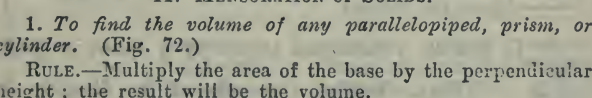
RULE.—The length of the arc of the evolute is equal to the difference between the lengths of the tangents drawn from either extremity of the arc to the involute.

II. MENSURATION OF SOLIDS.

1. *To find the volume of any parallelopiped, prism, or cylinder.* (Fig. 72.)

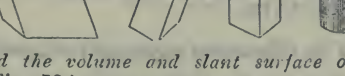
RULE.—Multiply the area of the base by the perpendicular height ; the result will be the volume.

FIG. 72.



2. *To find the volume and slant surface of a cone or pyramid.* (Fig. 73.)

FIG. 73.



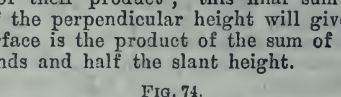
RULE.—Multiply the area of the base by $\frac{1}{3}$ the perpendicular height ; the product will be the volume. The slant

surface is equal to the perimeter of the base multiplied by half the slant height.

3. *To find the volume and slant surface of the frustum of a cone or pyramid.* (Fig. 74.)

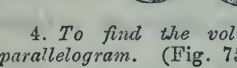
RULE.—To the sum of the areas of the two ends add the square root of their product; this final sum being multiplied by $\frac{1}{3}$ of the perpendicular height will give the volume. The slant surface is the product of the sum of the perimeter of the two ends and half the slant height.

FIG. 74.



4. *To find the volume of a wedge whose base is a parallelogram.* (Fig. 75.)

FIG. 75.



RULE.—Add the length of the edge to twice the length of the base; multiply the sum by the width of the base and the product by $\frac{1}{6}$ of the perpendicular height: the result will be the volume.

5. *To find the volume of a prismoid.* (Fig. 76.)

FIG. 76.

RULE.—To the sum of the areas of the two ends add four times the area of a section parallel to the base and equally distant from both ends; the sum being multiplied by $\frac{1}{6}$ the perpendicular height will give the volume.

6. *To find the volume and surface of a sphere or globe.* (Fig. 77.)

FIG. 77.

RULE.—Multiply the cube of the diameter by $\frac{\pi}{6}$ or ·5236; the product will be the volume. To obtain the surface, multiply the square of the diameter by π or 3·1416.

7. *To find the volume and surface of the segment of a sphere.* (Fig. 78.)

FIG. 78.

RULE.—Add the square of the height to 3 times the square of the radius of the base; that sum multiplied by the height and that product by $\frac{\pi}{6}$ or ·5236 will give the volume. To obtain the surface, multiply the diameter of the whole sphere by the height of the segment and that product by π or 3·1416.

8. *To find the volume and surface of a zone of a sphere.* (Fig. 79.)

RULE.—To the sum of the squares of the radii of the two ends add $\frac{1}{3}$ the square of the height; multiply the sum by the height and that result by $\frac{\pi}{2}$ or 1·5708: the result will be the volume. To obtain the surface, multiply the diameter of the whole sphere by the height of the zone, and that product by π or 3·1416.

FIG. 79.

9. *To find the volume and surface of a cylindrical ring.*

RULE.—To the thickness of the ring add the inner diameter; multiply that sum by the square of the thickness, and the product by $\frac{\pi^2}{4}$ or 2·4674: the result will be the volume. To obtain the surface, multiply the sum of the inner diameter and thickness by the thickness, and that product by π^2 or 9·87.

TABLE TO FIND THE VOLUME AND SURFACE OF ANY REGULAR POLYHEDRON.

V = volume. A = area. L = linear edge. r = radius of inscribed circle.

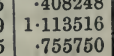
No. of Sides	No. of Edges in each side	Name	$A=L^2\times$	$V=L^3\times$	$r=L\times$
4	3	Tetrahedron	1·732051	·117851	·204124
6	4	Hexahedron[1]	6·000000	1·000000	·500000
8	3	Octahedron	3·464102	·471405	·408248
12	5	Dodecahedron	20·645729	7·663119	1·113516
20	3	Icosahedron	8·660254	2·181695	·755750

[1] Or cube.

10. *To find the volume of an ellipsoid.* (Fig. 80.)

RULE.—Multiply the product of the three principal axes by $\frac{\pi}{6}$ or ·5236: the result will be the volume.

FIG. 80.



11. *To find the volume of the segment of an ellipsoid of revolution when the base is circular.* (Fig. 81.)

RULE.—Take double the height of the segment from

Fig. 81.

three times the length of the fixed axis ; multiply the difference by the square of the height, and that product by $\frac{\pi}{6}$ or ·5236 : then that result multiplied by the square of the revolving axis and the product divided by the square of the fixed axis will give the volume.

12. *To find the volume of the segment of an ellipsoid of revolution when the base is elliptical.* (Fig. 82.)

Rule.—Take double the height of the segment from three times the length of the revolving axis ; multiply the difference by the square of the height, and that product by $\frac{\pi}{6}$ or ·5236 : then that result multiplied by the fixed axis, and the product divided by the revolving axis, will give the volume.

Fig. 82.

13. *To find the volume of the middle frustum of an ellipsoid of revolution when the ends are circular.* (Fig. 83.)

Fig. 83.

Rule.—Multiply the sum of the square of the middle diameter and one-half the square of the diameter of one end by the length of the frustum, and that product by $\frac{\pi}{6}$ or ·5236 for the volume.

14. *To find the volume of the middle frustum of an ellipsoid of revolution when the ends are elliptical.* (Fig. 84.)

Rule.—To twice the product of the transverse and conjugate diameters of the middle section, add the product of the transverse and conjugate diameters of one end ; multiply the sum by the height of the frustum, and that product by $\frac{\pi}{12}$ or ·2618 : the result will be the volume.

Fig. 84.

15. *To find the volume of a paraboloid.* (Fig. 85.)

Fig. 85.

Rule.—Multiply the square of the diameter of the base by the perpendicular height, and the result by $\frac{\pi}{8}$ or ·3927 ; the product will be the volume.

16. *To find the volume of the frustum of a paraboloid when its ends are perpendicular to its axis.* (Fig. 86.)

RULE.—Multiply the sum of the squares of the diameters of the two ends by the height of the frustum; the product multiplied by $\frac{\pi}{8}$ or ·3927 will be the volume.

FIG. 86.

17. *To find the volume of any solid of revolution.*

(I) The volume is represented by the definite integral $\int_o^h \pi y^2 . dx$, where Ox is the axis of symmetry.

RULE.—Divide the length of the axis into a convenient number of equal parts. Measure the ordinates, and treat their squares as if they were the ordinates of a plane curve of the same length as the solid; the area of this curve multiplied by π or 3·1416 will be the volume required.

(II) If the position of the centre of gravity of the generating area is known, the following method is applicable.

RULE.—Multiply the area of the generating section by the distance of its centre of gravity from the axis; 2π or 6·283 times the product will be the volume required.

Example.—Find the volume of the solid generated by the revolution of an equilateral triangle about its base.

If $2a$ be the side of the triangle, its area is $\sqrt{3} . a^2$; and its centre of gravity is $\frac{a}{\sqrt{3}}$ from the base. Hence the volume required is $2\pi \times \sqrt{3}a^2 \times \frac{a}{\sqrt{3}} = 2\pi a^3$.

18. *To measure the volume of any solid.*

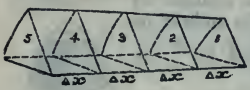
(I) *To measure the volume in slices.*

RULE.—Take one of the plane surfaces as the base, and divide the mass into slices parallel to that base and sufficiently thin as to be able either to neglect or account separately for the curvature.

Then take the volume of each slice separately, and add them together for the whole volume, taking account of the curvature in this addition if necessary.

(II) *To measure the volume by the rules applicable to the area of a plane curve.* (Fig. 87.)

FIG. 87.



RULE.—Take a straight line in the figure as a base line, or line of abscissa, and divide the figure along that line into any number of equal parts, and measure the areas of the plane sections at those points of division by the rules applicable to the area of a plane curve.

Then treat the areas thus found as if they were the ordinates of a plane curve of the same length as the figure, and the area of this will be the volume of the solid.

Example. (See fig. 87.)

Number of Sections	Areas of Sections in square feet	Multipliers	Products
1	5	1	5
2	10	4	40
3	15	2	30
4	20	4	80
5	25	1	25
			180
		$\frac{\Delta x}{3}$ =	2

Area = 360 cubic feet.

Remark.—The volume is above assumed to be represented by the definite integral $\int_0^h A \, . \, dx$, where A is the area of any section perpendicular to the base line ox. The volume may also be represented by the double integral $\iint z \, . \, dx \, . \, dy$ taken over the area of the base, the z axis being supposed perpendicular to the base.

(III) *To measure the volume by Dr. Woolley's method.* (Fig. 88.)

RULE.—Take a straight line in the figure as a base line, and divide the figure along that line by an odd number of parallel and equidistant planes perpendicular to the base. Then divide the figure horizontally in the same way by a number of plane sections parallel to the base. Then take ordinates at the intersections of the horizontal with the vertical plane sections in their consecutive order, and treat them as follows :—

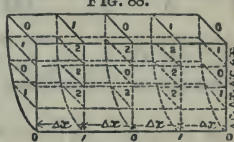
(1) Neglect absolutely all ordinates which are *odd* in *both* planes of section.

(2) Neglecting the outside rows of ordinates, double every ordinate which is *even* in *either* or *both* planes of section, and add them together.

(3) Add to this the simple sum of all the *even* ordinates in the outside rows.

(4) Multiply this final sum by $\frac{2}{3}$ of the product of the common vertical interval, by the common horizontal interval, and the result will be the volume.

FIG. 88.



Ex.: In the accompanying figure the multiplier for each ordinate is shown above it, so that if $s =$ the sum of the products of the ordinates by their respective multipliers, $v =$ the volume, and $\Delta x' =$ the common vertical interval, and $\Delta x =$ the common horizontal interval, then

$$v = \frac{2(s \times \Delta x' \times \Delta x)}{3}$$

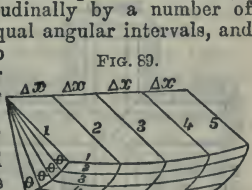
Remark.—This method is inferior in accuracy to that obtained by a double application of Simpson's rules.

19. *To measure the volume of a wedge-shaped solid bounded on one side by a curved surface.* (Fig. 89.)

The volume is represented by the double integral $\iint \frac{1}{2} r^2 . dx . d\theta$, where r is a radius from the edge. θ is the angle between the radius and the plane of the base, and ox is parallel to the edge.

RULE.—Divide the figure longitudinally by a number of planes radiating from the edge at equal angular intervals, and also divide the length of figure into a number of equal intervals for ordinates, and treat each of the radiating planes as follows :—

FIG. 89.



(I) Measure the ordinates as if for taking the areas of the several planes, but instead of the ordinates themselves compute their half-squares, and treat them as if they were the ordinates of a plane curve of the same length as the figure. The result of this calculation is called the moment of the radiating plane.

(II) Treat the moments of the radiating planes as if they were the ordinates of a curve, but taking the common angular interval in circular measure.

Example. (See fig. 89.)

No. of Planes	Moments of the Radiating Planes	Multipliers	Products
1	105	1	105
2	110	4	440
3	115	2	230
4	120	4	480
5	125	1	125
			1380

$$\frac{\theta}{3} = \frac{\text{angular interval}}{3} = \cdot 0291$$

1380
12420
2760
Volume = 40·1580

20. *To find the mean sectional area of a solid.*

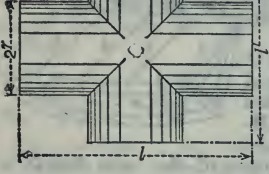
RULE.—Divide the volume of the solid by its length ; the result will be the mean sectional area.

21. *To set off the correct form of a mean cross-section.*

RULE.—Divide the figure longitudinally by a number of horizontal planes ; take the mean breadth of each of the horizontal planes and set them off perpendicular to a fixed straight line, and at the same height as their corresponding planes in the solid : a line passing through the ends of these mean breadths will be the correct form of the mean sectional area of the solid.

Note.—The mean breadth of a plane curve is found by dividing the area of the curve by its length.

FIG. 90.



22. *To find the volume of a fourway piece of piping.*

Let r (fig. 90) be the radius of the piping and l and l' the lengths.

Then volume $= \pi r^2 (l + l' - \frac{2}{3}r)$.

23. *To find the surface of any solid of revolution.*

(I) The surface is represented by the definite integral $\int_o^h \pi y \, . \, ds$, where ds is an element of arc of the generating curve.

RULE.—Divide the perimeter of the generating curve into a convenient number of equal arcs. Measure the ordinates at the points of division, and treat them as if they were the equidistant ordinates of a curve, with the common interval equal to the length of the arcs. The area of this curve multiplied by 2π or 6·283 will be the area of the surface.

(II) If the position of the centre of gravity of the periphery of the generating curve is known, the following method is applicable :—

RULE.—Multiply the length of the arc of the generating curve by the distance of the centre of gravity *of the arc* from the axis ; 2π or 6·283 times the product will be the area of the surface.

Example.—Find the surface of the solid generated by the revolution of an equilateral triangle about its base.

If $2a$ be the side of the triangle, its perimeter, exclusive of the axis, is $4a$; and the centre of gravity of the two sides is $\sqrt{3}a/2$ from the base.

Hence the surface required is $2\pi \times 4a \times \sqrt{3}\, a/2 = 4\sqrt{3}\, \pi a^2$.

24. *To find the area of any surface.*

(I) *Exact method.*—The area is given by the double integral $\iint \sec\theta \,.\, dx \,.\, dy$, where θ is the angle made by the surface with the xy plane. This is equal to $\iint \sqrt{(1 + \tan^2\phi + \tan^2\psi)} \,.\, dx \,.\, dy$, where ϕ and ψ are the angles made with the z axis by the sections of the surface with the xz and yz planes.

RULE.—Take a straight line in the figure as base line and divide the figure along that line by a convenient number of parallel and equidistant planes perpendicular to the base ; call these the vertical sections. Then divide the figure horizontally in the same way by a number of plane sections parallel to the base. At the intersections of the two sets of sections measure $\tan\phi$ and $\tan\psi$, ϕ and ψ being the angles made in the sections by the tangents to the curves with the base. Evaluate $\sqrt{1 + \tan^2\phi + \tan^2\psi}$ at each intersection. Treat this as the ordinate of a solid, and proceed to find its volume by any of the rules given above. The result is the area required. It is desirable that no part of the surface should be approximately perpendicular to the base.

Example.—A portion of a ship's side is bounded by two sections 40 feet apart, and two waterlines 8 feet apart. The tangents of the angles (ϕ) made with the middle line at the sections and at one situated midway between them, and the

tangents of the angles (ϕ) made with the middle line at the two waterlines, and at one midway, are as follows:—

W.L.	Section 1		Section 2		Section 3	
	$\tan\phi$	$\tan\psi$	$\tan\phi$	$\tan\psi$	$\tan\phi$	$\tan\psi$
1	·30	·21	·35	·16	·48	·06
2	·21	·22	·31	·17	·44	·07
3	·10	·22	·14	·18	·25	·09

nd $\sqrt{1+\tan^2\phi+\tan^2\psi}$ in each case, and proceed as in the table below :—

Simpson's Multiplier	Section 1 $\sqrt{1+\tan^2\phi+\tan^2\psi}$	Product	Section 2 $\sqrt{1+\tan^2\phi+\tan^2\psi}$	Product	Section 3 $\sqrt{1+\tan^2\phi+\tan^2\psi}$	Product
1	1·06	1·06	1·07	1·07	1·11	1·1
4	1·04	4·16	1·06	4·24	1·10	4·4
1	1·03	1·03	1·03	1·03	1·02	1·0
		7·25		6·34		6·5
		1		4		
		7·25	+	25·36	+	6·5
						= 39·1

The area of the curved surface is $39{\cdot}14 \times \frac{20}{3} \times \frac{4}{3} = 348$ square feet.

(II) *Approximate method.*

RULE.—Take girths along (say) the vertical sections at equidistant intervals. For each section in the half-breadth plan, note the angle at which the various waterlines cross, and estimate the mean slope of the waterlines surrounding the surface under question. The secant of this mean angle with the middle line is termed the modifying factor, and is multiplied by the girth concerned. These modified girths are then regarded as the ordinates of a curve, whose area is the surface required.

Example.—In the previous example, the girths at Sections 1, 2, and 3 are 8·2, 8·3, and 8·7 feet respectively. The respective mean angles of the waterlines with the middle line are ·22, ·17, and ·07. Find the surface.

The first modifying factor is $\sqrt{1+(\cdot22)^2}$ or 1·02 ; similarly the others are 1·015 and 1·0 approximately.

No.	Girth	Modifying Factor	Modified Girth	Simpson's Multiplier	Product
1	8·2	1·02	8·4	1	8·4
2	8·3	1·015	8·45	4	33·8
3	8·7	1·0	8·7	1	8·7
					50·9

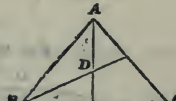
Area of surface $= 50{\cdot}9 \times \frac{20}{3} = 340$ square feet approximately.

CENTRES AND MOMENTS OF FIGURES.

To find the Centres of Gravity of a few Special Figures.

1. *Triangle.* (Fig. 91.)

Fig. 91.

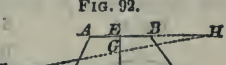


Rule.—From the middle points of any two sides draw lines to the opposite angle; the point of intersection D of these lines is the required centre.

Or, trisect the line joining the middle point of one side with the opposite vertex; the point of trisection nearer to the base is the required centre.

2. *Trapezoid.* (Fig. 92.)

Fig. 92.

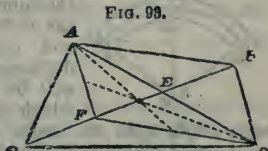


Rule.—Bisect AB in E and CD in F and join EF. Produce AB beyond B to H, making BH = CD, and produce CD beyond C to I, making CI = AB; then join HI, and where this line intersects EF is the centre of gravity G.

Note.—EG is to GF as $2CD + AB$ is to $2AB + CD$. If the angles at A and C are right angles, the distance of G from AC is equal to $\frac{AB^2 + AB \cdot CD + CD^2}{3\,(AB + CD)}$

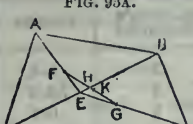
3. *Quadrilateral.* (Fig. 93.)

Fig. 93.



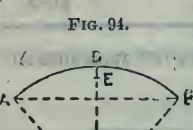
Rule.—Draw the diagonals AD and CB intersecting in E; along CB set off CF equal to EB, and join FA and FD; the centre of the triangle AFD will be the centre of the quadrilateral.

FIG. 93A.



Or, bisect the diagonal BD (fig. 93A) at E ; join EA, EC. Make EF = ⅓EA and EG = ⅓EC. Join FG, cutting BD at H. Make KG = HF. Then K is the required centre.

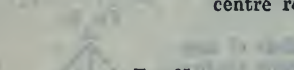
FIG. 94.



4. *Circular arc.* (Fig. 94.)

RULE.—Let ADB be the circular arc and C the centre of the circle of which it is a part ; bisect the arc AB in D, and join DC and AB ; multiply the radius CD by the chord AB, and divide by the length of the arc ADB ; lay off the quotient CE upon CD ; then E is the centre required.

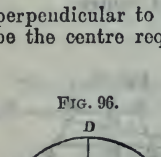
FIG. 95.



5. *Very flat curved line (approximate).* (Fig. 95.)

RULE.—Let ADB be the arc ; draw the chord AB, and bisect it in C; draw CD perpendicular to AB ; make CE equal to ⅔ of CD ; then E will be the centre required.

FIG. 96.

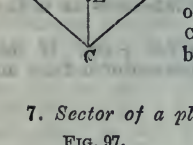


6. *Sector of a circle.* (Fig. 96.)

RULE.—Let ADC be the sector, E its centre; multiply the chord AB by ⅔ of the radius CA; divide the product by the length of the arc: the quotient equals the distance CE set along the line CD, D being at the bisection of the arc AB.

7. *Sector of a plane circular ring.* (Fig. 97.)

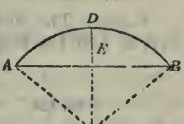
FIG. 97.



RULE.—Let CA be the outer and CE the inner radius of the ring; divide twice the difference of the cubes of the inner and outer radii by three times the difference of their squares; the quotient will be an intermediate radius CF, with which describe the arc FF, subtending the same angle with the sector : the centre H of the circular arc FF, found by Rule 4, will be the centre required.

8. *Circular segment.* (Fig. 98.)

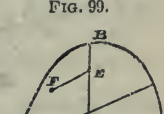
FIG. 98.



RULE.—Let C be the centre of the circle of which it is a part ; bisect the arc AB in D, and join CD ; divide the cube of half the chord AB by three times the area of half the segment ADB ; set off the quotient CE along CD, and E will be the centre required.

9. *Parabolic half-segment.* (Fig. 99.)

FIG. 99.



RULE.—Let ABD be a half-segment of a parabola, BD being part of a diameter parallel to the axis and AD an ordinate conjugate to that diameter—that is, parallel to a tangent at B. Make BE equal to $\frac{2}{3}$ BD, and draw EF parallel to AD and equal to $\frac{3}{8}$ AD. Then F will be the centre of the half-segment.

10. *Height of centre of semicircle or semi-ellipse from its base.*

RULE.—Multiply the radius of the semicircle (or that semi-axis of the ellipse which is perpendicular to the base) by 4, and divide the product by 3π.

11. *Height of centre of parabola from its base.*

RULE.—Multiply its vertical height by 2, and divide the product by 5.

12. *Prism or cylinder with plane parallel ends.*

RULE.—Find the centres of the ends ; a straight line joining them will be the axis of the prism or cylinder, and the middle point of that line will be the centre required.

13. *Cone or pyramid.*

RULE.—Find the centre of the base, from which draw a line to the summit ; this will be the axis of the cone or pyramid, and the point at $\frac{1}{4}$ from the base along that line will be the centre.

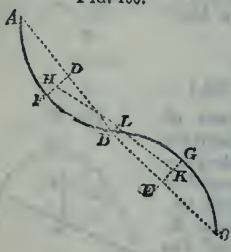
14. *Hemisphere or hemi-ellipsoid.*

RULE.—The distance of the centre from the circular or elliptic base is $\frac{3}{8}$ of the radius of the sphere, or of that semi-axis of the ellipsoid which is perpendicular to the base.

15. *Paraboloid.*

RULE.—The distance of its centre from the base along its axis is $\frac{1}{3}$ of the height from the base.

FIG. 100.



16. *To find the centre of gravity of any continuous curved line.* (Fig. 100.)

Ex.: Let ABC be the given curve; bisect it at B; join AB and BC, and bisect those chords at the points D and E respectively; set off FD perpendicular to AB, and EG perpendicular to BC; make $FH = \frac{1}{3}DF$ and $GK = \frac{1}{3}GE$, and join HK; bisect HK at the point L, which will be a close approximation to the position of the centre of gravity of the curved line ABC.

Remark.—If the line is too irregular to permit its two parts to be regarded as flat regular curves, it should be divided into four or eight parts as required. The points corresponding to L in the above figure are found separately for each pair of parts, joined in pairs and bisected; this process is repeated until only one point remains, this being the required centre of gravity.

RULES FOR FINDING THE MOMENTS AND CENTRES OF FIGURES.

The geometrical moment of a figure, whether a line, an area, or a solid, relatively to a given plane or axis is the product of the magnitude of that figure, into the perpendicular distance of its centre from the given plane or axis, and is equal to the sum of the moments of all its parts relatively to the same plane.

The centre of an area is determined when its distance from two axes in the plane of the figure is known.

The centre of a figure of three dimensions is determined

when its distance from three planes not parallel to one another is known.

1. *To find the moment of an irregular figure relatively to a given plane or axis.*

RULE.—Divide the figure into parts whose centres are known; multiply the magnitude of each of its parts into the perpendicular distance of its centre from the given plane or axis; distinguish the moments into positive and negative, according as the centres of the parts lie to one side or the other of the plane: the difference of the two sums will be the resultant moment of the figure relatively to the given plane or axis, and is to be regarded as positive or negative, according as the sum of the positive or negative moments is the greater.

2. *To find the perpendicular distance of the centre of an irregular figure from a given plane or axis.*

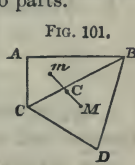
RULE.—Divide the moment of that figure relatively to the given plane or axis by its magnitude; the quotient will be the perpendicular distance of its centre from the given plane or axis.

3. *To find the centre of a figure consisting of two parts whose centres are known.* (Fig. 101.)

RULE.—Multiply the distance between the two known centres by the magnitude of either of the parts, and divide the product by the magnitude of the whole figure; the quotient will be the distance of the centre of the whole figure from the centre of the other part, the centre of the whole figure being in the straight line joining the centres of the two parts.

Ex.: Let ABCD be such a figure, M and m the magnitude of its two respective parts, $M + m$ the magnitude of the whole figure, D the distance between the centres M and m of the two parts, and C the centre of the whole figure.

FIG. 101.

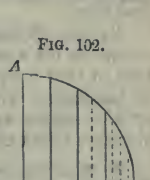


$$MC = \frac{m \times D}{M + m} \qquad mC = \frac{M \times D}{M + m}.$$

4. *To find the centre of any plane area by means of ordinates.* (Fig. 102.)

Let ABC, the quadrant of a circle, be such an area; CB the base line, divided into a number of equal parts by ordinates; AC the transverse axis traversing its origin.

FIG. 102.



1*st. Determine the perpendicular distance of the centre of the quadrant from the transverse axis in the following manner:—*

RULE.—Multiply each ordinate by its distance from the transverse axis; consider the products as ordinates of a new curve of the same length as the given figure: the area of that curve, found by the proper rule, will be the moment of the figure relatively to the transverse

axis; this moment, divided by the whole area of the figure, will give the perpendicular distance of its centre from the transverse axis.

In algebraical symbols the moment of a plane figure relatively to its transverse axis, and found by the above rule, is expressed thus:—

$$\int xydx.$$

Note.—In practice it is better to proceed as follows:—Multiply the ordinates first by their multipliers, and then those products by the number of intervals from the origin; take the sum of those products and multiply it by $\frac{1}{3}$rd of a whole interval squared, if Simpson's first rule is used, by $\frac{3}{8}$ths of a whole interval squared, if Simpson's second rule is used, and so on for the other rules.

Example.

No. of Intervals	Ordinates	Multipliers	Products	Products × No. of Intervals from Origin
0	16·0000	1	16·0000	·00000
1	15·4919	4	61·9676	61·9676
2	13·8564	$1\frac{1}{2}$	20·7846	41·5692
$2\frac{1}{2}$	12·4900	2	24·9800	62·4500
3	10·5830	$\frac{3}{4}$	7·93725	23·81175
$3\frac{1}{4}$	9·3274	1	9·3274	30·31405
$3\frac{1}{2}$	7·7460	$\frac{1}{2}$	3·8730	13·5555
$3\frac{3}{4}$	5·5678	1	5·5678	20·87925
4	0·0000	$\frac{1}{4}$	0·0000	·00000
		$\frac{\text{Interval}}{3}$	$\frac{150·43765}{\frac{4}{3}}$	$\frac{\text{Interval}^2}{3}$ = $\frac{254·54735}{\frac{16}{3}}$
	Approximate area = 200·58353			Approx. moment = 1357·585

$$\frac{\text{Moment } 1357·585}{\text{Area } 200·5835} = 6·768 \left\{ \begin{array}{l} \text{approximate perpendicular distance} \\ \text{of centre from the transverse axis.} \end{array} \right.$$

2nd. Find the perpendicular distance of its centre from the base line.

RULE.—Square each ordinate, and take the half-squares as ordinates for a new curve of the same length as the figure; the area of that curve, found by the proper rule, will be the moment of the figure relatively to the base line: this moment, divided by the whole area of the figure, will give the perpendicular distance of its centre from the base line.

In algebraical symbols the moment of a plane figure relatively to its base line, found by the above rule, is expressed thus:—

$$\int \frac{y^2}{2} dx.$$

Example.

No. of Intervals	Ordinates	Half-squares	Multipliers	Products
0	16·0000	128·0000	1	128·0000
1	15·4919	119·9995	4	479·9980
2	13·8564	95·9999	2	191·9998
3	10·5830	55·9999	4	223·9996
4	0·0000	0·0000	1	0·0000
				1023·9974

$$\frac{\text{Interval}}{3} = \frac{4}{3}$$

Approximate moment = 1365·3298

$$\frac{\text{Moment } 1365{\cdot}3298}{\text{Area } 201{\cdot}0624} = 6{\cdot}796 \left\{ \begin{array}{l} \text{approximate perpendicular dis-} \\ \text{tance of centre from base.} \end{array} \right.$$

Actual moment = 1365·3

Actual area = 201·0624

5. *To find the centre of a plane area bounded by a curve and two radii by means of polar co-ordinates.* (See fig. 68.)

1*st. Determine the perpendicular distance of its centre from a plane traversing the pole and at right angles to one of the bounding radii, called the first radius, in the following manner:—*

RULE.—Divide the angle subtended by the arc into a convenient number of equiangular intervals by means of radii; measure the lengths of the radii from the pole to the arc, and multiply the third part of the cube of each of them by the cosine of the angle which they respectively make with the first radius; treat these products by one of the rules applicable to finding the area of a plane curve (the only difference being that the common interval is taken in circular measure); the result will be the moment of the figure relatively to the plane traversing the pole: this moment, divided by the area of the figure, will give the perpendicular distance of its centre from the plane traversing the pole.

Example.

No. of Radii	Radii	Cubes of Radii / 3	Angles with First Radius	Cosines	Products	Simpson's Multipliers	Products
1	12	576	0°	1·0000	576·0000	1	576·0000
2	12	576	5°	·9962	573·8112	4	2295·2448
3	12	576	10°	·9848	567·2448	2	1134·4896
4	12	576	15°	·9659	556·3584	4	2225·4336
5	12	576	20°	·9397	541·2672	1	541·2672
							6772·4352

$$\frac{\text{Interval in circular measure}}{3} = \cdot0291$$

Moment relatively to plane traversing pole = 197·077864

$$\frac{\text{Moment } 197{\cdot}077864}{\text{Area } 25{\cdot}1327} = 7{\cdot}841 \left\{ \begin{array}{l} \text{perpendicular distance of centre} \\ \text{from plane traversing pole.} \end{array} \right.$$

In algebraical symbols the moment, as here found, is expressed thus:—

$$\int \frac{r^3}{3} \cos\theta d\theta.$$

2*nd. Determine the moment of the figure relatively to the first radius precisely in the same way as in the foregoing rule, with the exception that sines must be used in the place of cosines; this moment, divided by the area of the figure, will give the perpendicular distance of its centre from the first radius.*

Note.—It is usual, in practice, to defer the division of the cubes of the radii by 3 until after the addition of the products.

Example.

No. of Radii	Radii	Cubes of Radii / 3	Angles with First Radius	Sines of Angles	Products	Simpson's Multipliers	Products
1	12	576	0°	·0000	·0000	1	·0000
2	12	576	5°	·0872	50·2272	4	200·9088
3	12	576	10°	·1736	99·9936	2	199·9972
4	12	576	15°	·2588	149·0688	4	596·2752
5	12	576	20°	·3420	196·9920	1	196·9920
							1194·1732
						$\frac{\text{Interval in circular measure}}{3}$ =	·0291
						Moment relatively to first radius =	34·750440

$$\frac{\text{Moment } 34{\cdot}75044}{\text{Area } 25{\cdot}1327} = 1{\cdot}38 \left\{ \begin{array}{l} \text{perpendicular distance of centre from} \\ \text{first radius.} \end{array} \right.$$

In algebraical symbols the moment as here found is expressed thus:—

$$\int \frac{r^3}{3} \sin\theta d\theta.$$

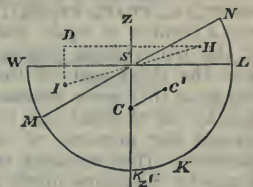
6. *To find the perpendicular distance of the centre of a solid, bounded on one side by a curved surface* (figs. 87 and 88), *from a plane perpendicular to a given axis at a given point.*

RULE.—Proceed as in Rule 4, p. 63, to find the moment relatively to the plane, substituting sectional areas for breadths: then divide the moment by the volume (as found by Rule 2, p. 54); the quotient will be the required distance. To determine the centre completely, find its distance from three planes no two of which are parallel.

7. *Having the moment and centre of a figure relatively to a given plane, to find the new moment and centre of the figure relatively to the same plane when a part of the figure is shifted.* (Fig. 103.)

In the figure WLK let C be its centre, and ZZ′ a plane with respect to which the moment of the figure is known; suppose the part WSM to be transferred to the new position SNL, so as to alter the shape of the figure from WLK to MNK; let I be the original and H the new centre of the shifted part: then *the moment of the figure* MNK *relatively to the plane* ZZ′ *is found as follows:—*

FIG. 103.



RULE.—Measure the distance, perpendicular to the plane of moments, between the centres of the original and new position of the shifted part, as HD, and multiply it by the magnitude of the shifted part; the product will be the moment required. *The new position of the entire figure is then found by the following rule:—*

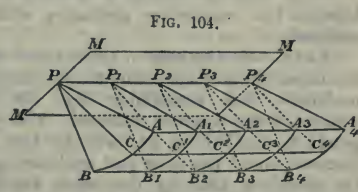
RULE.—Multiply the distance between the centres of the original and new position of the shifted part by the magnitude of that part; that product, divided by the magnitude of the whole figure, will give the distance the centre has traversed in the direction in which the part has been shifted, and in a plane parallel to a line joining the centres of the original and new position of the shifted part, as from C to C′ in fig. 103.

8. *To find the centre of a wedge-shaped solid* (fig. 104) *by means of polar co-ordinates.*

1*st. Determine the perpendicular distance of its centre relatively to a transverse sectional plane, as* PAB.

RULE.—Divide the solid by a number of parallel and equidistant planes, as PAB, $P_1A_1B_1$, $P_2A_2B_2$, &c.; then multiply each sectional area by its distance from the plane PAB; treat the products as though they were the ordinates of a curve of the same length as the figure; the area of that curve, found by the proper rule, will be the moment of the figure relatively to the plane PAB: that moment, divided by the volume of the figure, will be the distance required.

FIG. 104.



2nd. Determine the perpendicular distance of its centre relatively to a longitudinal plane passing through its edge, as MPM, *perpendicular to the first radius,* PB.

RULE.—Divide the figure by a number of longitudinal planes radiating from the edge MPM at equiangular intervals (as PP_1AA_1, PP_2CC_1, PP_3BB_1); also divide the length of the figure into a number of equal intervals by ordinates, and treat each of the longitudinal planes as follows:—Measure its ordinates, take the third part of their cubes, and treat those quantities as if they were ordinates of a new curve; that is, find its area by one of Simpson's rules: the area of that new curve is termed the moment of inertia of the longitudinal plane in question. Then multiply each moment of inertia of the several planes by the cosine of the angle made by the plane to which it belongs with the plane PB, and treat these products by a proper set of Simpson's multipliers; add together the products, and multiply the sum by $\frac{1}{3}$ of the common angular interval in circular measure if Simpson's first rule is used, and by $\frac{3}{8}$ if Simpson's second rule is used. The result will be the moment of the figure relatively to the plane MPM. This moment, divided by the volume of the figure, will be the distance required.

The algebraical expression for the moment as found in this rule is

$$\iint \frac{r^3}{3} \cos\theta \, dr \, d\theta.$$

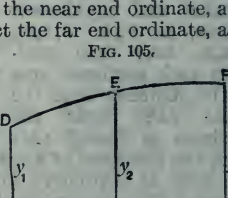
3rd. Determine the perpendicular distance of its centre relatively to a longitudinal plane passing through its edge, and a radius as PP^1BB^1, *by the foregoing rule, with the exception of multiplying by sines instead of cosines.*

Note.—In practice it is usual to defer the division of the cubes of the radii by 3 until after the addition of the products.

9. *To find the centre of gravity of a plane area contained between two consecutive ordinates, with respect to the near end ordinate.*

RULE.—To the sum of three times the near end ordinate, and ten times the middle ordinate, subtract the far end ordinate, and multiply the remainder by the square of the common interval. The product, divided by 24, will be the moment about the near end ordinate. On dividing this by the area, the longitudinal position of the centre of gravity is obtained.

FIG. 105.



Ex.: In fig. 105 let ABC be the base, and AD, BE, and CF the ordinates. Call them y_1, y_2, and y_3 respectively, and let the common interval be denoted by h. Then the moment of the area ABED about the

near end ordinate AD is equal to $\frac{(3y_1+10y_2-y_3)\times h^2}{24}$. If this be divided by the area of ABED (see p. 46), the quotient will be the distance of the C.G. from AD.

For an example, let the ordinates be 6·2, 8·5, and 9·4 feet, and the common interval 12 feet.

No. of Ordinates	Ordinates	Multipliers for Area	Products	Ordinates	Multipliers for Moments	Products
1	6·2	5	31·0	6·2	3	18·6
2	8·5	8	68·0	8·5	10	85·0
3	9·4	−1	−9·4	9·4	−1	−9·4
			89·6			94·2
		$\frac{\text{(Interval)}}{12}$ =	1		$\frac{\text{(Interval)}^2}{24}$ =	6
Area of portion included between 1 and 2			= 89·6	Moment about 1 = 565·2		

$\frac{\text{Moment}\ 565{\cdot}2}{\text{Area}\ 89{\cdot}6} = 6{\cdot}308$ { Perpendicular distance of centre of portion included between Nos. 1 and 2, from No. 1 ordinate.

Note.—When the moment of the area is required about the middle ordinate, the above multipliers should be changed to 7, 6, −1; so that moment $= \frac{7y_1 + 6y_2 - y_3}{24} \times h^2$.

Moments of Inertia and Radii of Gyration.

1. *To find the moment of inertia of a body about a given axis.*

Rule.—Conceive the body to be divided into an indefinitely great number of small parts; multiply the mass (or area) of each of these small parts into the square of its perpendicular distance from the given axis: the sum of all these products as obtained will be the moment of the body about the given axis.

2. *To find the square of the radius of gyration of a body about a given axis.*

Rule.—Divide the moment of inertia of the body relatively to the given axis by the mass (or area) of the body.

3. *Given the moment of inertia of a body about an axis traversing its centre of gravity in a given direction, to find its moment of inertia about another axis parallel to the first.*

RULE.—Multiply the mass (or area) of the body by the square of the perpendicular distance between the two axes, and to the product add the given moment of inertia.

4. *Given the separate moments of inertia of a set of bodies about parallel axes traversing their several centres of gravity, to find the moment of inertia of these bodies about a common axis parallel to their separate axes.*

RULE.—Multiply the mass (or area) of each body by the square of the perpendicular distance of its centre of gravity from the common axis; the sum of all these products, together with all the separate moments of inertia, will be the combined moment of inertia.

5. *Given the square of the radius of gyration of a body about an axis traversing its centre in a given direction, to find the square of the radius of gyration about another axis parallel to the first.*

RULE.—Square the perpendicular distance between the two axes, and add the product to the given square of the radius of gyration.

6. *To find the moment of inertia of a plane area, bounded on one side by a curve* (see fig. 102), *relatively to its base line.*

RULE.—Divide the base line into a suitable number of equal intervals, and measure ordinates at the points of division; take the third part of the cube of each of these ordinates, and treat those quantities so computed as the ordinates of a new curve: the area of that new curve, found by the proper rule, will be the moment of inertia required. In algebraical symbols the above rule is expressed thus:—

$$\int \frac{y^3}{3} dx.$$

Note.—When the moment of inertia is required as a whole, and not in separate parts, it is usual to postpone the division of the cubes till the end of the calculation.

7. *To find the moment of inertia of a plane area, bounded on one side by a curve, relatively to one of its ordinates.*

RULE.—Multiply each ordinate by its proper multiplier, according to one of the rules for finding the area of such figures; then multiply each of the products by the square of the number of whole intervals that the ordinate in question is distant from the

ordinate taken as the axis of moments: the sum of these products, multiplied by $\frac{1}{3}$ or $\frac{3}{8}$ the cube of a whole interval, according as Simpson's first or second rule is used, will be the moment of inertia required.

In algebraical symbols this rule is expressed thus:—

$$\int x^2 y dx.$$

Example I.

CALCULATION OF MOMENT OF INERTIA OF THE QUADRANT OF A CIRCLE RELATIVELY TO THE BASE LINE.

No. of Intervals	Ordinates	Cubes of Ordinates / 3	Multipliers	Products
0	16·00	1365·33	1	1365·33
1	15·49	1238·89	4	4955·56
2	13·86	887·50	$1\frac{1}{2}$	1331·25
$2\frac{1}{2}$	12·49	649·48	2	1298·96
3	10·58	394·76	$\frac{3}{4}$	296·07
$3\frac{1}{4}$	9·33	270·72	1	270·72
$3\frac{1}{2}$	7·75	155·16	$\frac{1}{2}$	77·58
$3\frac{3}{4}$	5·57	57·29	1	57·29
4	0·00	0·00	$\frac{1}{4}$	0·00
				9652·76
			Interval / 3	$= \frac{4}{3}$
				12870·34

Example II.

CALCULATION OF THE MOMENT OF INERTIA OF THE QUADRANT OF A CIRCLE RELATIVELY TO THE ENDMOST ORDINATE.

No. of Intervals	Ordinates	Multipliers	Products	Squares of Nos. of Intervals	Products
0	16·0000	1	16·0000	0·00	000
1	15·4919	4	61·9676	1·00	61·9679
2	13·8564	$1\frac{1}{2}$	20·7846	4·00	83·1384
$2\frac{1}{2}$	12·4900	2	24·9800	6·25	156·1250
3	10·5830	$\frac{3}{4}$	7·93725	9·00	71·4353
$3\frac{1}{4}$	9·3274	1	9·3274	10·5625	98·5207
$3\frac{1}{2}$	7·7460	$\frac{1}{2}$	3·8730	12·2500	47·4443
$3\frac{3}{4}$	5·5678	1	5·5678	14·0625	78·2972
4	9·0000	$\frac{1}{4}$	0·0000	16·0000	0·0000
					596·9288
				Interval^3 / 3	$= \frac{64}{3}$

Approximate moment of inertia = 12734·4810

Definition.—If a body be conceived divided into an infinite number of parts, and the mass (or area) of each part be multiplied by the square of its distance from a fixed point, the sum of all these products is termed the polar moment of inertia about the point.

8. *To find the polar moment of inertia of a plane area about a point.*

(I) RULE.—At equal angular intervals sufficient to include the whole area, draw radii from the point to the perimeter. Treat the fourth power of these radii as the ordinates of a new curve having a common interval equal to the angular interval between consecutive radii expressed in circular measure. One quarter of the area of this curve, found by the proper rule, is the polar moment of inertia required.

Example.—Find the polar moment of inertia of a semicircle of 5 feet radius about one end of the diameter.

The polar radii at an angular interval of 15° are 10·00, 9·66, 8·66, 7·07, 5·00, 2·59 feet.

No.	Radius	(Radius)^4	Multiplier	Product
1	10·00	10,000	1	10,000
2	9·66	8,735	4	34,940
3	8·66	5,624	2	11,248
4	7·07	2,498	4	9,992
5	5·00	625	2	1,250
6	2·59	45	4	180
7	—	—	1	—.
	Common interval = ·2618			67,610
				$\times \frac{1}{3} \times \frac{1}{2} \times$ ·2618

Polar moment of inertia = 1,475

(II) RULE.—If the moments of inertia about two perpendicular axes through the point are known, their sum is equal to the polar moment of inertia about the point.

Definitions.—The product of inertia of an area about two perpendicular axes is the algebraic sum of each element of area multiplied by the product of its co-ordinates with reference to the two axes. In the first and third quadrants the product of inertia is positive; in the second and fourth quadrants it is negative.

The principal axes of inertia through a point are those axes about which the product of inertia is zero.

9. *Given the moments and products of inertia about two perpendicular axes, to find the corresponding quantities about any two other perpendicular axes.*

RULE.—If O*x*, O*y* (fig. 106) are the axes, X and Y the moments of inertia about them, and P their product, the moments and

product of inertia about Ox, Oy' (denoted by X', Y', and P'), making a positive angle θ with the original axes, are given by the following formulæ:—

$$X' = X\cos^2\theta + Y\sin^2\theta - 2P\sin\theta\cos\theta,$$
$$Y' = X\sin^2\theta + Y\cos^2\theta + 2P\sin\theta\cos\theta,$$

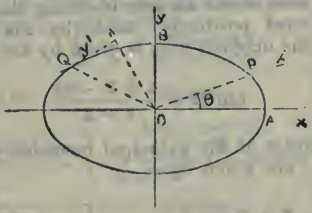
so that $X' + Y' = X + Y$; and

$$P' = P\cos 2\theta - \tfrac{1}{2}(Y - X)\sin 2\theta.$$

Note.—If Ox and Oy are principal axes, $P = 0$, and the formulæ become

$$X' = X\cos^2\theta + Y\sin^2\theta;\quad Y' = X\sin^2\theta + Y\cos^2\theta;$$
$$P' = -\tfrac{1}{2}(Y - X)\sin 2\theta.$$

FIG. 106.



If an ellipse (fig. 106) be drawn having its principal axes Ox, Oy along the principal axes of inertia, and of magnitude OA, OB equal to radii of gyration about Oy and Ox respectively, the radius of gyration about any other axis Ox' is represented by the perpendicular OM drawn to that tangent of the ellipse which is parallel to the axis Ox'; the moment of inertia about Ox' is proportional to the square of OM, or equally inversely proportional to the square of the radius OP along Ox'. The product of inertia about Ox', Oy' is similarly represented by the product of OM and MQ, where OQ is conjugate to OP.

10. *Given the moments and product of inertia about two perpendicular axes, to find the principal moments and axes of inertia.*

RULE.—If X, Y, and P are the moments and product of inertia respectively for the axes Ox, Oy, the angle θ (reckoned positively) made by the principal axes Ox', Oy', with the original axes is given by the formula—

$$\tan 2\theta = \frac{2P}{Y - X}$$

The magnitudes of the principal moments of inertia X′, Y′ about Ox′, Oy′, are given by—

$$X' = \frac{Y+X}{2} - \sqrt{\frac{(Y-X)^2}{4} + P^2}; \quad Y' = \frac{Y+X}{2} + \sqrt{\frac{(Y-X)^2}{4} + P^2};$$

assuming X′ to be the least and Y′ the greatest moment of inertia.

11. *Given the moments of inertia about three axes, two perpendicular and one bisecting the angle between them, to find the principal moments and axes of inertia.*

RULE.—If X, Y, are the moments of inertia about the axes Ox, Oy, and Z that about an axis bisecting the angle yOx, the angle θ (reckoned positively) made by the principal axes Ox′, Oy′, with the original axes is given by the formula—

$$\tan 2\theta = \frac{Y+X-2Z}{Y-X}$$

The magnitudes of the principal moments of inertia X′, Y′ about Ox′, Oy′, are given by—

$$X' = \frac{Y+X}{2} - \sqrt{Z^2 - Z(Y+X) + \frac{Y^2+X^2}{2}}$$

$$Y' = \frac{Y+X}{2} + \sqrt{Z^2 - Z(Y+X) + \frac{Y^2+X^2}{2}}$$

Note.—Since $X+Y = X'+Y'$, the sum of the moments of inertia about any two perpendicular axes is constant.

If the area has an axis of symmetry, the principal axes are along and perpendicular to this axis.

Ex.—An unequal-sided parallelogram is formed of two right-angled isosceles triangles of 1 inch side. Find the principal moments and axes of inertia.

Take Ox parallel to the shorter sides, and Oz perpendicular to the longer sides. Then $X = \frac{1}{12}$; $Y = \frac{1}{6}$; $Z = \frac{5}{24}$.

By the formulæ above $\tan 2\theta = -2$;

$\theta = -32°$ or $58°$, the former corresponding to the least moment of inertia.

$$\text{Greatest I or } Y' = \tfrac{1}{8} + \frac{\sqrt{5}}{24} = \cdot 218.$$

$$\text{Least I or } X' = \tfrac{1}{8} - \frac{\sqrt{5}}{24} = \cdot 032.$$

TABLE OF SQUARES OF RADII OF GYRATION OF A FEW SPECIAL FIGURES.

Body	Axis	Radius^2 =
Rectangle; sides a and b	side a	$\frac{b^2}{3}$
	axis through C.G. parallel to side a	$\frac{b^2}{12}$
Square; side a	any axis through C.G.	$\frac{a^2}{12}$
Triangle; sides a, b, c; heights a', b', c'	side a	$\frac{a'^2}{6}$
	axis through C.G. parallel to side a	$\frac{a'^2}{18}$
Equilateral triangle; height d	any axis through C.G.	$\frac{d^2}{18}$
Trapezoid; height h, parallel sides a and b	side a	$\frac{h^2}{6} \cdot \frac{a+3b}{a+b}$
	axis through C.G. parallel to side a	$\frac{h^2}{18} \cdot \frac{a^2+4ab+b^2}{(a+b)^2}$
Trapezoid with two right angles; parallel sides a and b, perpendicular side h	side h	$\frac{1}{6}(a^2+b^2)$
	axis through C.G. parallel to side h	$\frac{a^4+2a^3b+2ab^3+b^4}{18(a+b)^2}$
Circle; diameter a	diameter	$a^2/16$
	centre (polar)	$a^2/8$
Ellipse; diameters a, b	diameter a	$\frac{b^2}{16}$
Common parabola; height a, base b perpendicular to axis	axis of parabola	$2b^2/5$
	base b	$8a^2/35$
	axis through C.G. parallel to base b	$12a^2/175$
Sphere; radius r	diameter	$\frac{2r^2}{5}$
	centre (polar)	$\frac{3r^2}{5}$
Spherical shell; external and internal radii r_1 and r_2	diameter	$\frac{2(r_1^5-r_2^5)}{5(r_1^3-r_2^3)}$
Ellipsoid of revolution; transverse semi-axis r	axis of revolution	$\frac{2r^2}{5}$
Ellipsoid; semi-axes a, b, c	axis $2a$	$\frac{b^2+c^2}{5}$
Circular cylinder; radius r, length $2a$	longitudinal axis	$\frac{r^2}{2}$
	transverse diameter through C.G.	$\frac{r^2}{4}+\frac{a^2}{3}$
Hollow circular cylinder; radius—external r_1, internal r_2; length $2a$	longitudinal axis	$\frac{r_1^2+r_2^2}{2}$
	transverse diameter through C.G.	$\frac{r_1^2+r_2^2}{4}+\frac{a^2}{3}$
Elliptic cylinder; semi-axes b, c, length $2a$	longitudinal axis	$\frac{b^2+c^2}{4}$
	transverse axis $2b$ through C.G.	$\frac{c^2}{4}+\frac{a^2}{3}$
Cone; height h, radius of base r	longitudinal axis	$\frac{3}{10}r^2$
	transverse axis through C.G.	$\frac{3}{80}h^2+\frac{3}{20}r^2$
	transverse axis through base	$\frac{h^2}{10}+\frac{3}{20}r^2$
	plane of base	$\frac{h^2}{10}$

Moment of Inertia = square of radius of gyration × mass (or area) of the figure.

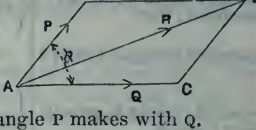
MECHANICAL PRINCIPLES.

Resultant and Resolution of Forces.

1. *To find the resultant of two forces acting through one point but not in the same direction.* (Fig. 107.)

Let AB, AC represent the two forces P and Q acting through the point A; complete the parallelogram ABCD: then its diagonal AD will represent in magnitude and direction the resultant of the two forces P and Q.

Fig. 107.



R = resultant. θ = angle P makes with Q.

a = angle R makes with Q. β = angle R makes with P.

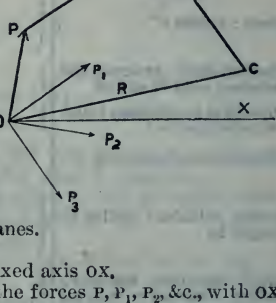
$$R = \sqrt{P^2 + Q^2 + 2.P.Q.\cos\theta};$$

$$\sin a = \sin\theta \frac{P}{R}; \quad \sin\beta = \sin\theta \frac{Q}{R}.$$

2. *To find the resultant of any number of forces acting in the same plane and through one point but not in the same direction.* (Fig. 108.)

Let P, P_1, P_2, P_3 be the forces acting through the point of application O; commence at O and construct a chain of lines OP, PA, AB, BC, representing the forces in magnitude and parallel to them; let C be the end of the chain: then a line R joining OC will represent in magnitude and direction the resultant of the forces P, P_1, P_2, and P_3.

Fig. 108.



Note.—This *geometrical* problem is true whether the forces act in the same or in different planes.

R = resultant.

θ = angle made by R with a fixed axis OX.

a, a_1, a_2, &c. = angles made by the forces P, P_1, P_2, &c., with OX.

ΣX = sum of the series of $P \cdot \cos a + P_1 \cdot \cos a_1 + P_2 \cdot \cos a_2$, &c.

ΣY = sum of the series of $P \cdot \sin a + P_1 \cdot \sin a_1 + P_2 \cdot \sin a_2$, &c.

$$R \cdot \cos\theta = \Sigma X. \qquad R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$$

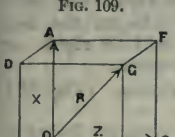
$$R \cdot \sin\theta = \Sigma Y. \qquad \tan\theta = \frac{\Sigma Y}{\Sigma X}$$

$$\cos\theta = \frac{\Sigma X}{R}$$

$$\sin\theta = \frac{\Sigma Y}{R}.$$

3. *To find the resultant of three forces acting through one point and making right angles with one another.* (Fig. 109.)

FIG. 109.



Let OA, OB, OC represent in magnitude and direction the forces X, Y, Z acting through one point O; complete the rectangular solid AEFB: then its diagonal OG will represent in magnitude and direction the resultant of the forces X, Y, Z.

R = resultant.

α, β, γ = the angles R makes with X, Y, Z, respectively.

$$Y = R \cdot \cos \beta. \quad R = \sqrt{X^2 + Y^2 + Z^2}.$$
$$Z = R \cdot \cos \gamma. \quad X = R \cdot \cos \alpha.$$

4. *To find the resultant of any number of forces acting through one point in different directions and not in the same plane.*

Let P, P_1, P_2, &c., be the forces α, β, γ; $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$, the angles their directions make with three axes passing through the point of application and making right angles with one another.

R = resultant.

$$\Sigma X = P \cdot \cos \alpha + P_1 \cdot \cos \alpha_1 + P_2 \cdot \cos \alpha_2 + \&c.$$
$$\Sigma Y = P \cdot \cos \beta + P_1 \cdot \cos \beta_1 + P_2 \cdot \cos \beta_2 + \&c.$$
$$\Sigma Z = P \cdot \cos \gamma + P_1 \cdot \cos \gamma_1 + P_2 \cdot \cos \gamma_2 + \&c.$$

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}$$

$$\cos \alpha = \frac{\Sigma X}{R}$$

$$\cos \beta = \frac{\Sigma Y}{R}$$

$$\cos \gamma = \frac{\Sigma Z}{R}.$$

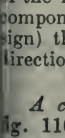
N.B. Cosines of obtuse angles are negative.

Note.—P cos α, P cos β, and P cos γ are termed the components of the forces in the directions of X, Y, and Z respectively. The components of the resultant are obtained by adding (allowing for sign) the components of the several forces in their respective directions.

PARALLEL FORCES.

A couple consists of two equal forces, as P and Q (see fig. 110), acting in parallel and opposite directions to one another, and is termed a right- or left-handed couple, according to whether the forces tend to turn in a clockwise direction or the reverse.

FIG. 110.



The moment of a couple is the product of either of the forces into the perpendicular distance AB between the lines of direction of the forces. The distance AB is termed the arm or lever of the couple.

5. *To find the resultant moment of any number of couples acting upon a body in the same or parallel planes.*

RULE.—Add together the moments of the right- and left-handed couples separately; the difference between the two sums will be the resultant moment, which will be right- or left-handed, according to which sum is the greater.

6. *To find the resultant of two parallel forces.* (Fig. 111 and 112.)

The magnitude of the resultant of two parallel forces is their sum of difference, according to whether they act in the same or contrary directions.

FIG. 111.

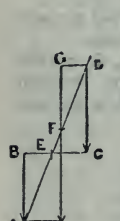
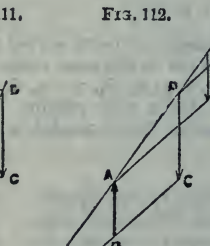


FIG. 112.



Let fig. 111 represent a case in which the two forces act in the same direction, and fig. 112 a case in which the components act in opposite directions.

Let AB and CD represent two forces; join AD and CB, cutting each other in E; in DA (produced in fig. 112) take DF =BA; through F draw a line parallel to the components; this will be the line of the resultant, and if two lines DG and AH be drawn parallel to BC, cutting the line of action of the resultant in G and H, GH will represent the magnitude of the resultant.

Or, numerically, the line of action of the resultant is obtained by adding (allowing for sign) the moments of the two forces about any point, this being equal to the moment of the resultant; the perpendicular distance of the line of action from the point is obtained by dividing this moment by the magnitude of the resultant.

$$AF = \frac{DC \cdot AD}{GH} \qquad DF = \frac{AB \cdot AD}{GH}$$

7. *To find the resultant of any number of parallel forces.*

RULE.—Take the sum of all those forces which act in *one* direction, and distinguish them as positive; then take the sum of all the other forces which act in the *contrary* direction, and distinguish them as negative. The direction of the resultant (positive or negative) will be in that of the greater of these two sums, and its magnitude will be the difference between them.

8. *To find the position of the resultant of any number of parallel forces when they act in two contrary directions.*

RULE.—1st. Multiply each force by its perpendicular distance from an assumed axis in a plane perpendicular to the

lines of action of the forces ; distinguish those moments into right- and left-handed, and take their resultant, which divide by the resultant force : the quotient will be the perpendicular distance of that force from the assumed axis.

2nd. Find by a similar process the perpendicular distance of the resultant force from another axis perpendicular to the first and in the same plane.

9. *To find the resultant of any number of couples not necessarily in a plane.*

Two couples of equal moments in the same or in parallel planes are equivalent to one another, whatever the magnitudes and positions of the forces composing the couples may be. A couple is therefore conveniently represented by a line perpendicular to its plane, and of length proportional to its moment ; usually the direction of the line is taken so that its relation to the direction of the couple is the same as that between the travel and the rotation of a right-handed screw. Note that any two parallel lines of the same magnitude and sense represent the same couple.

RULE.—Replace the couples by lines as above, giving them their correct magnitudes and direction, and treat these as forces through a point by Rule 4. The resultant gives the magnitude and direction of the resultant couple.

10. *To find the resultant of any number of forces in a plane.*

RULE.—Treat them as forces through any fixed point by Rule 2, and find their resultant. Calculate also the moment of each force about the point, and add them together allowing for the sign of each. The resultant moment divided by the magnitude of the resultant force gives the perpendicular distance of its line of action from the point.

Definition.—The moment of a force about a line that it does not meet is the product of the component of the force perpendicular to the line with the shortest distance between the line and the line of action of the force.

11. *To find the resultant of any number of forces, not in one place.*

RULE.—Resolve the forces parallel to three perpendicular axes as in Rule 4, and find the magnitude and direction of their resultant R. Calculate the moments of each component about the three axes, and treating these as couples find the resultant couple F by Rule 9. Resolve this couple into couples G parallel to the force R, and H perpendicular to R. Resolve the couple H into 2 forces R_1, R_2, of which R_1 is equal and opposite to R, while R_2 is equal and parallel to R ; find the position of R_2 (not in plane of figure). Then the final resultant is equal to the force R_2 combined with the couple G (since R and R_1 neutralize). The combination of a force in and a couple about the same line is termed a *wrench*.

CENTRE OF GRAVITY.

1. *To find the moment of a body's weight relatively to a given plane.*

RULE.—Multiply the weight of the body by the perpendicular distance of its centre of gravity from the given plane.

2. *To find the common centre of gravity of a set of detached bodies relatively to a given plane.*

RULE.—Find their several moments relatively to a fixed plane; take the algebraical sum or resultant of those moments and divide it by the total sum of all the weights: the quotient will be the perpendicular distance of the common centre of gravity from the given plane.

Note.—When the moments of some of the weights lie on one side of the plane, and some on the other, they must be distinguished into positive and negative moments, according to the side of the plane on which they lie, and the difference between the two sums of the positive and negative moments will be the resultant moment. The sign of the resultant will show on which side the common centre of gravity lies.

Let w, w^1, w^2, &c. = the several weights.

d, d^1, d^2, &c. = the several perpendicular distances of the centres of gravity of w, w^1, w^2, &c., from the plane of moments.

D = the perpendicular distance of their common centre of gravity from the plane of moments.

$$D = \frac{wd + w^1d^1 + w^2d^2 + \&c.}{w + w^1 + w^2 + \&c.}$$

3. *To find the centre of gravity of a body consisting of parts of unequal heaviness.*

RULE.—Find separately the centre of gravity of these several parts, and then treat them as detached weights by the foregoing rule.

4. *To find the distance through which the common centre of gravity of a set of detached weights moves when one of those weights is shifted into a new position.*

RULE.—multiply the weight moved by the distance through which its centre of gravity is shifted; divide the product by the sum total of the weights: the quotient will be the distance through which the common centre of gravity has moved in a line parallel to that in which the weight was shifted.

Let w = weight shifted.

d = distance through which w was moved.

W = sum total of weights.

D = distance through which the common centre of gravity has moved in a line parallel to that in which the shifted weight was moved.

$$D = \frac{wd}{W}; \quad d = \frac{DW}{w}.$$

MOTION.

Velocity.

The speed of a body or of a point within a body is the distance travelled in an infinitesimal space of time divided by that time. The velocity of the body takes also into account the direction in which the body is moving and is completely represented by a line drawn in the direction of motion, whose length represents to scale the speed.

Composition of velocities.—To combine several velocities impressed simultaneously upon a body, if OP, OP_1, OP_2, OP_3 (fig. 108, p. 76) represent the component velocities, draw PA parallel and equal to OP, AB, and BC parallel and equal respectively to OP_2 and OP_3. OC is the resultant velocity of the body. Similarly the resultant velocity OC may be resolved into two component velocities in any required directions X and Y by drawing lines from *od*, *dc* parallel to X and Y; the lengths *od*, *dc* represent the magnitudes of the component velocities.

Example.—If a boat is propelled at a speed and in a direction represented by AC (fig. 107, p. 76) in a stream whose velocity is represented by AB, the resultant velocity of the boat is represented by AD. To combine any number of velocities analytically, resolve each along three axes at right angles (or two if all the velocities are in one plane) by multiplying each velocity by the cosine of the angle which it makes with the axis; add, allowing for sign, the components along each direction. The sums are the components of the resultant velocity in the three directions, which may be compounded as above. E.g., if v_1, v_2, v_3, . . . are the velocities making angles α_1, α_2, α_3, . . . with the axis Ox, β_1, β_2, β_3 . . . with the axis Oy, and γ_1, γ_2, γ_3, . . . with the axis Oz, the components P, Q, R, of the resultant along Ox, Oy, Oz, are given by—

$$P = v_1 \cos \alpha_1 + v_2 \cos \alpha_2 + v_3 \cos \alpha_3 + \ldots$$
$$Q = v_1 \cos \beta_1 + v_2 \cos \beta_2 + v_3 \cos \beta_3 + \ldots$$
$$R = v_1 \cos \gamma_1 + v_2 \cos \gamma_2 + v_3 \cos \gamma_3 + \ldots$$

The resultant S is given by $S^2 = P^2 + Q^2 + R^2$; and it makes angles A, B, C, with the axis, given by—

$$\cos A = \frac{P}{S}; \quad \cos B = \frac{Q}{S}; \quad \cos C = \frac{R}{S}$$

Velocity diagram for a linked mechanism.—To find the velocity of any part of a linked mechanism, a velocity diagram may be drawn as illustrated in the following example. AOB represents diagrammatically (fig. 113) the crosshead of a screw-steering gear, AC, BD, the connecting links, and C and D are forced by guides to follow the axis of the frame OO. If the velocity of A is known, that of C (or any other part) can be found; and conversely.

Draw *oa* to represent the velocity of A, *oa* being perpendicular to OA. *ob* in the opposite direction represents that of B. The velocity of C relative to A is necessarily perpendicular to AC, while relative to the frame it is parallel to the axis. Therefore, draw *ca* perpendicular to CA, and *oc* parallel to the axis; this gives *c*. Similarly the point *d* is obtained. *oc* and *od* are the velocities of the points C and D. The velocity of any other point, say E in the connecting link AC, is obtained by dividing *ac* at *e* so that *ae* : *ec* = AE : EC. Join *oe*, which is the velocity of the point E.

FIG. 113.

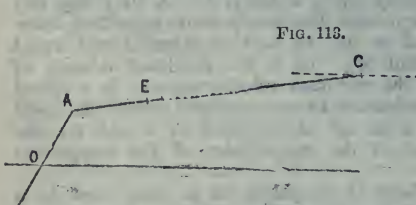
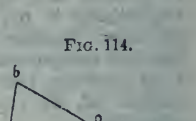
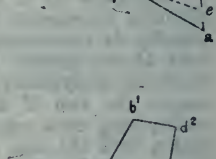


FIG. 114.





If *f* be the middle point of *cd*, *of* is the mean velocity of C and D, i.e. the velocity of the screw shaft as a whole, to allow for which a small amount of play has to be given. Note that the shaft is moving towards the crosshead, and that the velocities of C and D relative to the shaft are given by *fc*, *df*.

ANGULAR VELOCITY.

The angular velocity of a body about an axis is the angle turned through about the axis in an infinitesimal space of time divided by that time. It is usually expressed in radians per second or in revolutions per minute, the unit in the former case being $\frac{60}{2\pi}$ or 9·55 times that in the latter.

Composition and resolution of angular velocities.—The angular velocity about an axis may be represented by a line

drawn parallel to the axis, and of length proportional to the magnitude of the angular velocity. The direction of the line usually bears the same relation to the direction of rotation as that existing between the travel and rotation of a right-handed screw. When so represented, angular velocities are combined and resolved in the same way as linear velocities (see p. 81).

ACCELERATION.

The acceleration of a body is the rate of change of its velocity or the change of velocity in an infinitesimal space of time divided by that time. The velocity of a body comprises both its speed and its direction; hence the acceleration may generally be divided into two parts—(*a*) that due to increase of speed, which is represented by $\frac{dv}{dt}$, where v is the speed, and is tangential to the direction of motion; (*b*) that due to alteration of direction, which is directed normally towards the centre of curvature and is equal to v^2/ρ where ρ is the radius of curvature of the path.

Angular acceleration is the rate of increase of angular velocity; and, for a body revolving about a fixed axis, is represented by $\frac{d\omega}{dt}$ where ω is the angular velocity.

Composition of accelerations.—Accelerations, linear and angular, are combined and resolved similarly to velocities.

Acceleration diagram for a linked mechanism.—To find the acceleration of any part of a linked mechanism, a velocity diagram is first constructed as in fig. 113, p. 82. To find the accelerations of the steering gear shown, that of A, assuming the crosshead to revolve uniformly, is equal to $\frac{v^2}{\rho}$ or $\frac{oa^2}{OA}$, and may be represented by $o'a'$ parallel to AO. The acceleration $o'b'$ of OB is equal and opposite. The normal acceleration of C relative to A is represented by $a'c^2$ equal to $\frac{ac^2}{AC}$ and drawn parallel to CA; the tangential acceleration of C relative to A is represented by c^2c' drawn perpendicular to c^2a', the length of c^2c' being at present unknown. The acceleration of C relative to the frame OO is parallel to the axis; so that $o'c'$ is drawn parallel to the axis meeting c^2c' at c', giving the length of $o'c'$. Similarly the acceleration of D is found by drawing $b'd^2$, and d^2d', giving $o'd'$. That of any point E in the link AC is obtained by joining $a'c'$ and dividing it at e', so that $a'e : e'c' = AE : EC$. Join $o'e'$, which is the acceleration of the point E. The accelerations of C and D relative to the screw are equal to $\frac{1}{2}c'd'$.

DYNAMICS.

Relations between Force and Motion.

Ox, Oy, Oz = 3 perpendicular axes.
P, Q, R = component forces along Ox, Oy, Oz acting on a body.
u, v, w = component velocities along Ox, Oy, Oz.
m = mass of body.
M_x, M_y, M_z = momenta parallel to Ox, Oy, Oz respectively = mu, mv, mw.
f, g, h = component accelerations parallel to Ox, Oy, Oz.
g = acceleration due to gravity = 32·2 in foot-second units.

$$P = mf = m\frac{du}{dt} = \frac{dM_x}{dt} = m\frac{d^2x}{dt^2}; \quad Q = mg = m\frac{dv}{dt} = \frac{dM_y}{dt} = m\frac{d^2y}{dt^2};$$

$$R = mh = m\frac{dw}{dt} = \frac{dM_z}{dt} = m\frac{d^2z}{dt^2}$$

Note.—In the above, if m is in pounds, P, Q, R are in poundals, one poundal being equal $\frac{1}{g}$ pound or about half an ounce weight; if, on the other hand, the forces are expressed in pounds, the mass m must be expressed in terms of the gravitational unit equal to g (about 32) pounds.

Example.—A force of 2 lb. acts upon a mass of 3 lb. To find the acceleration.

The mass in gravitational units $= \frac{3}{g}$

$\therefore$ Acceleration $= \frac{P}{m} = \frac{2}{3/g} = \frac{2g}{3} = 21\frac{1}{3}$ ft. per second per second.

Angular Motion.

I = mass moment of inertia about axis of revolution.
ξ = angular acceleration.
ω = angular velocity.
θ = angle turned through.
M = angular momentum = $I\omega$.
G = moments of forces about axis.

$$G = I\xi = I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt^2} = \frac{dM}{dt}$$

Note.—If G is expressed in foot-pounds, I must be expressed in the gravitational unit, or is I/g of the density of the material multiplied by the volume moment of inertia of the body (see pp. 69-75).

Work and Energy.

The work done by a force on a body is the product of the force by the distance moved resolved along the direction of the force. The work done by a couple is the moment of the couple

multiplied by the angle turned through resolved along the plane of the couple. With the previous notation, if the body runs through distances x, y, z parallel to the axes, and rotates through an angle θ, the work done is $\text{P}x + \text{Q}y + \text{R}z + \text{G}\theta$, allowing for sign.

The energy of a body is its capacity for doing work.

Kinetic energy is energy due to motion. With the preceding notation, its amount is $\frac{1}{2g} m (u^2+v^2+w^2) + \frac{1}{2g} \text{I}\omega^2$, m and I being expressed in pounds and the result in foot-pounds.

Potential energy is energy due to position, and is measured from an arbitrarily fixed datum. A body of height h feet above the sea-level has potential energy of mh foot-pounds. A ship has potential energy due both to the height of its centre of gravity and the depth of its centre of buoyancy.

Molecular energy, due to heat, electrical state, magnetism, vibration, etc., is frequently waste energy as far as its capacity of doing useful work is concerned.

Conservation of energy. The work done on a body (other than that involved in a change in the potential energy) in a given interval of time is equal to the increase of its total energy.

Power is the rate of doing work. It is equal to $\text{P}u + \text{Q}v + \text{W}w + \text{G}\omega$, allowing for sign. This is equal to the rate of increase of energy. The practical unit of power is the *horse-power*, equivalent to 550 foot-pounds per second, or 33,000 foot-pounds per minute. Another unit is the *watt*, 746 of which are equivalent to one H.P.

Uniform Force in Line of Motion.

P = uniform force in pounds weight.
m = mass in pounds.
f = uniform acceleration = $\text{P}g/m$.
V = initial velocity in feet per second.
s = distance travelled.
t = time occupied.
v = final velocity.
$v = \text{V} + ft$; $v^2 = \text{V}^2 + 2fs$; $s = \text{V}t + \frac{1}{2}ft^2$.
For retarded motion change f to $-f$.
For motion vertically under gravity f to g or $-g$, according as the initial motion is downwards or upwards. In that case $\text{P} = \pm m$.
For motion down an incline of angle α to the horizontal, replace f by $g \tan \alpha$.
For angular rotation with the notation above, Ω being the initial angular velocity, $\omega = \Omega + \xi t$; $\omega^2 = \Omega^2 + 2\xi\theta$; $\theta = \Omega t + \frac{1}{2}\xi t^2$.

Gravity.

g = acceleration due to gravity in feet/second^2.
λ = latitude of the place.
h = height above sea-level.

R = radius of earth in feet = 20,900,000.

$g = 32{\cdot}088\ (1 + {\cdot}005302\ \sin^2\phi - {\cdot}000007\ \sin^2 2\phi - \frac{2h}{R})$.

Usually g is taken as 32·2, or 981 in centimetres/second^2.

Simple Vibration.

M = mass in pounds.
a = semi-amplitude of vibration.
n = frequency or number of double vibrations per second.
E = modulus or force in pounds required to produce unit extension.
t = time.
x = displacement at time 't.'
f = acceleration at time 't.'
α = a constant.
$g = 32{\cdot}2$.

$$n = \frac{1}{2\pi}\sqrt{\frac{Eg}{M}};\quad x = a\sin(2\pi nt + \alpha);\quad f = -\frac{Eg}{M}x.$$

Simple Pendulum.

L = length of pendulum in feet.
T = time of a single small vibration in seconds.
g = acceleration due to gravity = 32·2.

$$T = \pi\sqrt{\frac{L}{g}} = {\cdot}554\sqrt{L}.$$

Table giving the Lengths of Pendulums in Inches that Vibrate Seconds in various Latitudes.

Sierra Leone	39·01997	New York	39·10120
Trinidad	39·01888	Bordeaux	39·11296
Madras	39·02630	Paris	39·12877
Jamaica	39·03503	London	39·13907
Rio Janeiro	39·04350	Edinburgh	39·15504

Table giving the Times of Vibration for Pendulum swinging through Large Arcs.

Angle swung on each side of vertical	30°	60°	90°	120°	150°	180°
Actual time of vibration ÷ Time for infinitely small angle	1·017	1·073	1·183	1·373	1·762	Infinite

Compound Pendulum.

κ = radius of gyration of body about axis of rotation.
h = centre of gravity below axis.
l = length of equivalent pendulum.

$$l = k^2/h.$$

The *centre of percussion*, or point at which a blow struck perpendicularly to the axis will cause no stress at the axis, is situated at a distance l (determined by the above formula) below the axis.

Centrifugal Force.

F = centrifugal force of body revolving in a circle at a uniform speed, or apparent force required to balance that necessary to produce the requisite normal acceleration.
W = weight of body.
N = number of revolutions per minute.
n = number of revolutions per second.
v = linear velocity in feet per second.
ω = angular velocity in circular measure per second.
r = radius of circle in feet.
g = acceleration due to gravity = 32·2 nearly.

$$F = \frac{Wv^2}{gr} = \frac{Wr\omega^2}{g} = \frac{4Wn^2\pi^2r}{g} = \frac{Wn^2r}{\cdot 8154} = \frac{WN^2r}{2935}$$

Gyroscopic Action.

If the axis of a revolving body is made to rotate into a new position, resistance is experienced due to the 'gyroscopic action' of the revolving mass. Let AB represent in the usual way the angular momentum $I\omega$ of a body having a moment of inertia I about the axis of revolution, and an angular velocity ω. If this axis is forced to occupy after a short time the position AC, BC represents the change of angular momentum. This is equal to $I\omega \times \angle BAC$. If this change is effected by turning the axis uniformly with angular velocity ω', the rate of change of angular momentum is $I\omega\omega'$, which is equal to the moment G of the applied couple. Note that the plane of G is perpendicular to that of shaft rotation, and of the direction of movement. If I is in weight units (lbs. $\times$ feet^2), and N and N′ are the number of revolutions per minute of shaft rotation, and of bodily rotation,

$$G = \frac{I}{g} \times \frac{4\pi^2 NN'}{3600} = \frac{I.N.N'}{2935}$$

In the case of a ship going ahead with a right-handed screw, the forces required on the shaft when turning to starboard are downward aft and upward forward; the reaction on the hull is then such as to cause a slight trim by the bow.

Impact.

u_1, u_2 = the velocities of two bodies before impact (if moving in opposite directions make u_2 negative).

v_1, v_2 = the velocities after impact.

m_1, m_2 = the masses.

e = coefficient of restitution = ratio of velocity of separation to that of approach.

For direct impact,

$$v_1 = \frac{u_1(m_1 - em_2) + m_2 u_2 (1 + e)}{m_1 + m_2}$$

$$v_2 = \frac{u_1 m_1 (1 + e) + u_2 (m_2 - em_1)}{m_1 + m_2}$$

$$\text{Kinetic energy lost} = \frac{m_1 m_2 (u_1 - u_2)^2 (1 - e^2)}{2g (m_1 + m_2)}$$

Total momentum is unchanged, or $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$.

For oblique impact, resolve the velocities along and perpendicular to the line of impact; treat the components along the line by the above formulæ; the latter are unaltered by the impact.

The value of the coefficient e depends to some extent on the shape of the bodies and the velocity of impact, as well as on the material. Approximate values for the impact of like materials are given in the following table:—

Material	Cast Iron	Mild Steel	Soft Brass	Lead	Elm	Glass	Ivory
e	·70	·67	·38	·20	·60	·94	·81

HYDROSTATICS.

The *density* of a fluid is the weight of a unit volume. Generally it is stated in pounds per cubic foot, or inversely as the number of cubic feet required to weigh 1 ton. (See tables on p. 262.)

The *specific gravity* of a fluid is the ratio of its density to that of water.

Density of a Mixture of Two Liquids.

w_1, w_2 = densities of the two liquids.

w = density of the mixture.

m_1, m_2 = proportion of the two liquids in the mixture by volume.

n_1, n_2 = proportion of the two liquids in the mixture by weight.

$$n_1/n_2 = m_1 w_1/m_2 w_2; \quad w = \frac{m_1 w_1 + m_2 w_2}{m_1 + m_2} = \frac{n_1 + n_2}{\frac{n_1}{w_1} + \frac{n_2}{w_2}}$$

Pressure in a Liquid.

w = density of liquid in pounds per cubic foot.
z = depth below free surface.
p = intensity of pressure in pounds per square inch.
P = intensity of pressure in pounds per square foot.
$P = wz$; $p = wz/144$.

In salt water, $w = 64$, $P = 64z$, $p = \frac{4}{9}z$.

In fresh water, $w = 62{\cdot}5$, $P = 62{\cdot}5z$, $p = {\cdot}433z$.

If the *absolute* pressure be required P and p must be increased by 2120 and 14·7 respectively, in order to allow for the pressure of the atmosphere.

Note.—The *centre of pressure* of an immersed plane surface is that point on the surface through which the resultant pressure acts.

Pressure on Immersed Plane Surface.

If surface be vertical find the centre of gravity G and take axes Gx horizontal and Gy vertically downwards. Let

A = area of plane.
h = depth of centre of gravity below free surface.
w = density of fluid.
T = total thrust or pressure on plane.
$\bar{x}$, $\bar{y}$ = co-ordinates of the centre of pressure.

Then $T = wAh$

$$\bar{y} = \frac{1}{Ah}\int y^2 \, . \, dx \, . \, dy \text{ over area.}$$

$$= \frac{1}{Ah} \times \text{moment of inertia of area about G}x.$$

$$\bar{x} = \frac{1}{Ah}\int xy \, dx \, dy \text{ over area.}$$

$$= \frac{1}{Ah} \times \text{product of inertia of area about G}x, \text{G}y.$$

If the surface and the axis Gy be inclined at an angle θ to the vertical, T and $\bar{x}$ are unaltered, but the value found for $\bar{y}$ should be multiplied by $\cos\theta$.

Pressure on any Closed Surface.

The resultant pressure on the whole immersed surface of a body is equal to the weight of the water displaced by the body and acts vertically upwards through the centre of gravity of the displaced volume. The upward force is termed the *displacement*, and the point through which it acts the *centre of buoyancy*.

DISPLACEMENT, Etc.

Computation of a Ship's Displacement.

This consists of computing the volume of the body of the vessel below the water-plane, up to which it is required to know her displacement, by one of the rules used for finding the volume of solids bounded on one side by a curved surface (see pp. 54, 55).

Two processes are generally made use of in computing a vessel's displacement, as the calculations in each process are required to determine the position of the centre of gravity of displacement, or centre of buoyancy, and also because the two results are a check on the correctness of the calculations.

One process consists in dividing the length of the ship on the load water-line by a number of equidistant vertical sections, computing their several areas by one of Simpson's rules, and then treating them as if they were the ordinates of a new curve, the base of which is the load water-line.

The other process consists in dividing the depth of the vessel below the load water-line by a number of equidistant horizontal planes parallel to the load water-line; the areas of their several planes are then computed by one of Simpson's rules, and those areas are treated as if they were the ordinates of a new curve, the base of which is the vertical distance between the load water-line and lowest horizontal plane.

As the vessel generally consists of two symmetrical halves, the volume of only half the vessel, below the load water-line, is calculated, the ordinates all being measured from a longitudinal vertical plane at the middle of the ship.

Usually the portion below the lowest water-line is treated, as are also the stern, rudder, bilges, keels, etc., as an appendage, its volume being calculated by means of equidistant vertical sections. The water-lines that are 'snubbed' or cut short abaft the fore perpendicular or before the after perpendicular are conceived to extend to these perpendiculars, the extra volumes thus introduced being regarded as negative appendages.

The displacement of a ship can also be obtained by dividing the length into sections, spaced as required by Tchebycheff's rule; the integration in a longitudinal direction is effected by simple summation. The water-lines are equidistantly spaced and integrated by Simpson's rules as before. This method is, generally speaking, more expeditious than is the one previously described, since fewer ordinates can be employed, and half the multiplication is dispensed with.

Both methods are illustrated in the displacement sheets given on pp. 94 ff.

Determination of a Ship's Centre of Buoyancy for the Upright Position.

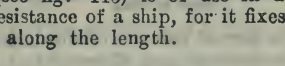
The centre of buoyancy is also termed the centre of gravity of displacement, as it occupies the same point as the centre of gravity of the volume of water displaced by the vessel, and its position is determined by the rules used for finding the centre of gravity of solids, bounded on one side by a curved surface (see rules, pp. 66 and 67), with the exception that its position need only be determined for its vertical distance from a horizontal plane, and its horizontal distance from a vertical plane; for the ship consisting of two symmetrical halves, it must necessarily lay in the longitudinal vertical plane in the middle of the ship.

Calculation of the centre of buoyancy is generally performed on the displacement sheet (see pp. 94 ff.).

Curve of Areas of Sections.

This curve (see fig. 115) is of use in designing and in estimating the resistance of a ship, for it fixes the distribution of displacement along the length.

Fig. 115.



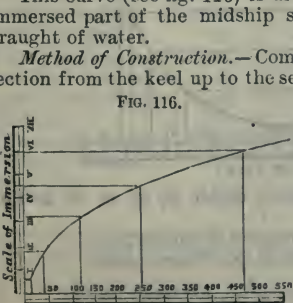
Method of Construction.—Compute the area of each transverse section up to the L.W.L.; and set it off to scale on a base of length. A curve drawn through the tops of the ordinates will form the curve required.

Curve of Areas of Midship Section.

This curve (see fig. 116) is used to determine the area of the immersed part of the midship section of a vessel at any given draught of water.

Method of Construction.—Compute the areas of the midship section from the keel up to the several longitudinal water-planes which are used for calculating the displacement; set these areas off along a base line as ordinates, in their consecutive order, the abscissæ of which represent to scale the respective distances between the longitudinal water-planes: a curve bent through the extremities of these ordinates will form the required curve.

Fig. 116.

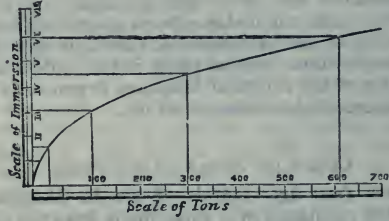


Curve of Displacement.

This curve is used to determine the displacement a vessel has at any draught of water parallel to the load water-line (see fig 117).

Method of Construction.—This curve is constructed in a similar manner to the foregoing curve, with the exception that the ordi-

Fig. 117



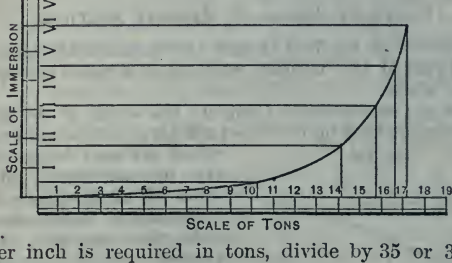
nates represent the several volumes of displacement (in tons of 35 cubic feet for salt water, and 36 cubic feet for fresh water) up to their respective longitudinal water-planes.

Curve of Tons per Inch of Immersion.

This curve (see fig. 118) is used to determine the number of tons required to immerse a vessel one inch at any draught of water parallel to the load water-plane.

To find the displacement per inch in cubic feet at any water-plane, divide the area of that plane by 12; and if the displace-

Fig. 118.



ment per inch is required in tons, divide by 35 or 36, as the case may be.

A = area of longitudinal water-plane in square feet.

T = tons per inch of immersion at that water-plane.

$$T = \frac{A}{12 \times 35} \text{ for salt water; } T = \frac{A}{12 \times 36} \text{ for fresh water.}$$

Method of Construction.—This curve is also constructed in a similar manner to the two foregoing curves, with the exception that the ordinates represent to scale the tons per inch of immersion at the respective water-planes.

The coefficients of fineness of a vessel consist of the block coefficients (β), the prismatic coefficient (γ), and the midship section coefficient (μ). They are determined from the following equations:—

V = volume of displacement in cubic feet.

L = length of vessel at load water-line in feet (or length between perpendiculars, according to convention).

B = extreme immersed breadth in feet. (Occasionally this is taken as the breadth at L.W.L. in cases where this is less than the extreme breadth.)

D = mean draught of water in feet. (Take to top of keel if bar keel.)

Σ = Area of midship section up to L.W.L. in square feet.

$$\beta = \frac{V}{L.B.D.}\ ;\ \gamma = \frac{V}{L\Sigma}\ ;\ \mu = \frac{\Sigma}{B.D.}\ ;\ \beta = \gamma \cdot \mu.$$

Another coefficient sometimes used is that of water-line area (λ) given by $\lambda = \frac{A}{L.B.}$ where A is the area of the L.W.L. Usually this is expressed as a coefficient $\frac{L.B.}{\text{tons per inch}}$, this latter being equal to $\frac{420}{\lambda}$

Values of these four coefficients for typical ships are given in the table below.

TABLE OF COEFFICIENTS OF FINENESS.

Class of Ship	Block Coefficient $\beta=\frac{V}{LBD}$	Prismatic Coeffic't $\gamma=\frac{V}{L\Sigma}$	Mid. Sec. Coeffic't $\mu=\frac{\Sigma}{BD}$	Waterline Coeffic't $\lambda=\frac{A}{LB}$	$\frac{LB}{\text{Tons per Inch}}$
Battleship (modern) .	·60	·62	·965	·73	575
Battleship (older) .	·65	·68	·95	·81	520
First-class Cruiser .	·56	·62	·90	·68	620
Modern Light Cruiser .	·58	·63	·92	·76	550
Torpedo Boat Destroyer	·55	·67	·82	·76	550
Steam Yacht . .	·52	·565	·92	·69	610
Fast Passenger Steamer	·59	·62	·95	·70	600
Large Cargo Vessels .	·73	·77	·95	·83	510
Sailing Yacht . .	·2	·5	·4	·75	560
Tug	·58	·61	·95	·76	550

Note.—The 'length' in warships is the length between perpendiculars.

Length between perpendiculars, 385 feet; breadth, 41 feet; draught at perpendicula

Simpson's multipliers	Appendage below lowest Water-line: Half Areas	Functions of Areas	Levers	Functions of Moments about 11	C.G. of Sections below 7 W.L.	Functions of Moments about 7 W.L.	WATER-LINES: 7 W.L.	6 W.L.	5 W.L.	4 W.L.	3 W.L.
							SIMPSON'S MULTIPLIERS: ½	2	1	2	1
	—	—	10	—	—	—	—	— —	·8 ·4	·8 ·4	·5 ·5
2	—	—	9	—	—	—	—	1·76 3·52	2·6 5·2	3·0 6·0	3·1 3·1	6
1	·9	·9	8	7·2	·3	·27	1·6 ·8	1·6	3·5 7·0	3·5	4·66 4·66	4·66	5·3 10·6	5·3	5·7 5·7	5
2	2·05	4·1	7	28·7	·45	1·84	3·0 1·5	6·0	5·5 11·0	11·0	6·94 6·94	13·88	7·9 15·8	15·8	8·54 8·54	17
1	3·67	3·67	6	22·02	·48	1·76	4·6 2·3	9·2	7·64 15·28	7·64	9·4 9·4	9·4	10·5 21·0	10·5	11·2 11·2	11
2	5·24	10·48	5	52·4	·51	5·34	6·4 3·2	12·8	9·9 19·8	19·8	11·7 11·7	23·4	12·86 25·72	25·72	13·66 13·66	27
1	7·18	7·18	4	28·72	·54	3·87	8·46 4·23	8·46	12·04 24·08	12·04	13·9 13·9	13·9	15·04 30·08	15·04	15·8 15·8	15
2	9·88	19·36	3	59·28	·57	11·25	10·3 5·15	20·6	14·1 28·2	28·2	15·86 15·86	31·72	16·9 33·8	33·8	17·6 17·6	35
1	12·48	12·48	2	24·96	·6	7·5	11·96 5·98	11·96	15·6 31·2	15·6	17·34 17·34	17·34	18·4 36·8	18·4	19·0 19·0	19
2	14·3	28·6	1	28·6	·63	18·0	13·1 6·55	26·2	16·54 33·08	33·08	18·26 18·26	35·52	19·3 38·6	38·6	19·9 19·9	39
1	15·65	15·65	0	251·88	·66	10·33	13·6 6·8	13·6	17·1 34·2	17·1	18·7 18·7	18·7	19·7 39·4	19·7	20·26 20·26	20
2	15·7	31·4	1	31·4	·66	20·7	13·46 6·73	26·92	17·0 34·0	34·0	18·74 18·74	37·48	19·64 39·28	39·29	20·24 20·24	40
1	14·2	14·2	2	28·4	·65	9·22	12·5 6·25	12·5	16·2 32·4	16·2	18·06 18·06	18·06	19·2 38·4	19·2	19·9 19·9	19
2	11·35	22·7	3	68·1	·67	14·8	10·6 5·3	21·2	14·8 29·6	29·6	17·1 17·1	34·2	18·46 36·92	36·92	19·3 19·3	38
1	8·1	8·1	4	32·4	·69	5·58	8·3 4·15	8·3	12·8 25·6	12·8	15·4 15·4	15·4	17·0 34·0	17·0	18·1 18·1	18
2	5·72	11·44	5	57·2	·71	8·14	5·9 2·95	11·8	10·4 20·8	20·8	13·2 13·2	26·4	15·1 30·2	30·2	16·4 16·4	32
1	3·9	3·9	6	23·4	·73	2·88	3·7 1·85	3·7	7·3 14·6	7·3	10·3 10·3	10·3	12·64 25·28	12·64	14·34 14·34	14
2	2·63	5·26	7	36·82	·75	3·94	2·2 1·1	4·4	4·34 8·68	8·68	6·9 6·9	13·8	9·5 19·0	19·0	11·6 11·6	23
1	·54	·27	8	2·16	·3	·081	1·0 ·5	1·0	2·1 4·2	2·1	3·8 3·8	3·8	6·0 12·0	6·0	8·4 8·4	8
2	—	—	9	—	—	—	·76 ·38	1·52	·8 1·6	1·6	1·5 1·5	3·0	3·0 6·0	6·0	5·0 5·0	10
½	—	—	10	—	—	—	·76 ·38	·38	·76 1·52	·38	·76 ·76	·38	·76 1·52	·38	1·5 1·5	1
		200·0		279·88 251·88 28·00		125·47	197·54 98·77 6	+ 284·94 569·83 5	+ 337·94 337·94 4	+ 375·88 751·76 3	+ 404 404
							592·62	+ 2849·40	+ 1351·76	+ 2255·28	+ 808

N.B.—The dark figures are the ordinates; the light figures under them and also to their right are t
oducts of the ordinates by their respective Simpson's multipliers, which are placed at the head a
the left of the table; if each row and column of these products be added together, and the resul

DISPLACEMENT, ETC., USING SIMPSON'S FIRST RULE.

3 feet for'd, 14 feet aft, 13 ft. 6 in. mean; waterlines apart, 2 feet; ordinates apart, 19 ft. 3 in.

2 W.L.		1 W.L.		VERTICAL SECTIONS				METACENTRES								
									Transverse		Longitudinal					
2		1		Functions of Areas	Multiples of Areas	Multipliers for Leverage	Moments	Ordinates of Load Water-line	Cubes of Ordinates	Functions of Cubes	Functions of Ordinates	Multipliers	Functions for C. of G. of W. Plane	Multipliers	Functions for Moment of Inertia of W. Plane	
·2	·1	—	—	3·3	1·65	10	16·5	—	—	—	—	10	—	10	—	
·16	6·32	3·3	6·6													
·32		1·65		23·19	46·38	9	417·42	3·3	36	72	6·6	9	59·4	9	534·6	
·04	6·04	6·34	6·34													
·08		3·17		44·01	44·01	8	352·08	6·34	255	255	6·34	8	50·72	8	405·76	
·94	17·88	9·26	18·52													
·88		4·63		66·29	132·58	7	928·06	9·26	793	1586	18·52	7	120·64	7	907·48	
·66	11·66	11·94	11·94													
·32		5·97		88·47	88·47	6	530·82	11·94	1702	1702	11·94	6	71·64	6	429·84	
·14	28·28	14·46	28·92													
·28		7·23		109·59	218·18	5	1095·9	14·46	3023	6046	28·92	5	144·60	5	723·0	
·26	16·26	16·54	16·54													
·52		8·27		128·88	128·88	4	515·52	16·54	4525	4525	16·54	4	66·16	4	264·64	
·0	36·0	18·2	36·4													
·0		9·1		145·71	291·42	3	874·26	18·2	6029	12058	36·4	3	109·2	3	327·6	
·3	19·3	19·36	19·36													
·6		9·68		158·6	158·6	2	317·2	19·36	7256	7256	19·36	2	38·72	2	77·44	
·14	40·28	20·2	40·4													
·28		10·1		166·77	333·54	1	333·54	20·2	8242	16484	40·4	1	40·4	1	40·4	
·46	20·46	20·5	20·5													
·92		10·25		170·53	170·53	0	5381·3	20·5	8615	8615	20·5	0	710·48	0	0	
5	41·0	20·5	41·0													
0		10·25		170·24	340·48	1	340·48	20·5	8615	17230	41·0	1	41·0	1	41·0	
·24	20·24	20·3	20·3													
·48		10·15		165·64	165·64	2	331·28	20·3	8365	8365	20·3	2	40·6	2	81·2	
7	39·9	19·9	39·8													
4		9·95		157·57	315·14	3	945·42	19·9	7881	15762	39·8	3	119·4	3	358·2	
7	18·7	19·04	19·04													
4		9·52		144·17	144·17	4	576·68	19·04	6902	6902	19·04	4	76·16	4	304·64	
2	34·4	17·76	35·52													
4		8·88		126·83	253·66	5	1268·3	17·76	5600	11200	35·52	5	177·6	5	888·0	
46	15·46	16·24	16·24													
92		8·12		105·41	105·41	6	632·46	16·24	4283	4283	16·24	6	97·44	6	584·64	
14	26·28	14·3	28·6													
28		7·15		80·71	161·42	7	1129·94	14·3	2924	5848	28·6	7	200·2	7	1401·4	
56	10·56	12·1	12·1													
12		6·05		56·07	56·07	8	448·56	12·1	1772	1772	12·1	8	96·8	8	774·4	
1	14·2	9·0	18·0													
2		4·5		33·18	66·36	9	597·24	9·0	729	1458	18·0	9	162·0	9	1458·0	
24	1·62	5·16	2·58													
48		2·58		14·74	7·37	10	73·7	5·16	138	69	2·58	10	25·8	10	258·0	
				424·48	438·7		6344·06			131488	438·7		1037·00		9860·24	
				848·96	219·35		5381·30						710·48			
				1			962·76						326·52			
				848·96	8706·78											

(Continued on next page.)

...grated by the proper multipliers, and the sums of these products added together, the two sums agree if the calculations are correct. In this case the sum thus obtained by two methods is 8281'0.

Displacement – main solid $= \frac{3231 \cdot 0}{35} \times \frac{8}{9} \times 19 \cdot 25 \times 2 = 3159$ tons.

Moment below L.W.L. – main solid $= \frac{8706 \cdot 78}{35} \times \frac{8}{9} \times 19 \cdot 25 \times 4 = 17,026$ ft.-tons.

Moment abaft ⋈ – main solid $= 962 \cdot 76 \times \frac{8}{9} \times (19 \cdot 25)^2 \times 2 = 18,121$ ft.-tons.

Displacement – lower appendage $= \frac{200}{35} \times \frac{4}{3} \times 19 \cdot 25 = 147$ tons.

C.G. below 7 W.L. – lower appendage $= \frac{125 \cdot 47}{200} = \cdot 63$ feet.

Moment abaft ⋈ – lower appendage $= \frac{28}{35} \times \frac{4}{3} \times (19 \cdot 25)^2 = 395$ ft.-tons.

Item.	Tons.	Below L.W.L.		Abaft ⋈	
		Distance.	Moment.	Distance.	Moment.
Main solid	3159		17026		18121
Lower appendage	147	12·63	1857		395
Aft ,,	5·7	1·3	7	197·8	1127
Fore ,,	·6	6·0	4	– 193·6	– 116
Rudder	2·2	8·7	19	194·3	427
Bilge keels	·6	9·5	6	—	—
Shafting	3·2	6·3	20	153·8	492
Shaft brackets	1·2	6·5	8	168·5	202
Propellers	·9	6·7	6	173·1	156
Swell	·6	5·8	3	121·5	73
Recess	– 2·8	5·9	– 16	146·6	– 410
Negative appendage aft	– 7·0	11·2	– 78	176·0	– 1232
Total	3311·2	5·70	18862	5·8	19235

Displacement 3311 tons; C.B. 5·70′ below L.W.L., 5·8′ abaft ⋈.

Area of L.W.L. (main portion) $= 438 \cdot 7 \times \frac{4}{3} \times 19 \cdot 25 = 11,260$ sq. ft.

Moment abaft ⋈ $= 326 \cdot 52 \times \frac{4}{3} \times (19 \cdot 25)^2 = 161,300$ ft.^{3}

Moment of inertia about ⋈ $= 9860 \cdot 24 \times \frac{4}{3} \times (19 \cdot 25)^3 = 93,700,000$ ft.^{4}

Item.	Area.	C.F. abaft ⋈	Moment about ⋈	Moment of Inertia about ⋈
Main portion	11,260		161,300	93,700,000
Appendage aft	100	198	19,800	3,900,000
Total	11,360	15·95	181,100	97,600,000

$11,360 \times (15 \cdot 95)^2 = 2,900,000$

Moment of inertia about C.F. $= 94,700,000$ ft.^{4}

Tons per inch $= \frac{11,360}{420} = 27{\cdot}0$. C.F. abaft ⋇ $= 15{\cdot}95'$.

Longitudinal BM $= \frac{94,700,000}{3311 \times 35} = 818'$. BG (with G in L.W.L.) $= 6'$ approx.

∴ Longitudinal GM $= 818 - 6 = 812$.

Moment to change trim 1 inch $= \frac{3311 \times 812}{12 \times 385} = 582$ ft.-tons.

Transverse BM $= \frac{131,488}{3311 \times 35} \times \frac{4}{3} \times \frac{1}{3} \times 19{\cdot}25 = 9{\cdot}78$ ft.

Area of midship section $= 170{\cdot}5 \times \frac{4}{3} \times 2 = 454$ sq. ft.

		'A.' W.L. (2 ft. above L.W.L.)				2 W.L.			3 W.L.		
No.	S.M	Ordinates.	Functions of Ordinates.	Cubes.	Functions of Cubes.	Ordinates.	Cubes.	Functions of Cubes.	Ordinates	Cubes.	Functions of Cubes.
1	½	—	—	—	—	·2	—	—	·5	—	—
2	2	3·45	6·90	41	82	3·16	31	62	3·1	30	60
3	1	6·65	6·65	294	294	6·04	220	220	5·7	185	185
4	2	9·55	19·10	871	1742	8·94	717	1634	8·54	623	1246
5	1	12·2	12·20	1816	1816	11·66	1581	1581	11·2	1405	1405
6	2	14·65	29·30	3144	6288	14·14	2833	5666	13·66	2549	5098
7	1	16·7	16·70	4657	4657	16·26	4291	4291	15·8	3944	3944
8	2	18·25	36·50	6078	12156	18·0	5832	11664	17·6	5452	10904
9	1	19·3	19·30	7189	7189	19·3	7189	7189	19·0	6859	6859
10	2	20·05	40·10	8060	16120	20·14	8181	16362	19·9	7881	15762
11	1	20·4	20·40	8490	8490	20·46	8552	8552	20·26	8316	8316
12	2	20·4	40·80	8490	16980	20·5	8615	17230	20·24	8292	16584
13	1	20·2	20·20	8242	8242	20·24	8300	8300	19·9	7881	7881
14	2	19·85	39·70	7821	15642	19·7	7645	15290	19·3	7189	14378
15	1	19·1	19·10	6968	6968	18·7	6539	6539	18·1	5930	5930
16	2	18·0	36·00	5832	11664	17·2	5088	10176	16·4	4411	8822
17	1	16·65	16·65	4616	4616	15·46	3690	3690	14·34	2950	2950
18	2	14·95	29·90	3341	6682	13·14	2274	4548	11·6	1561	3122
19	1	13·05	13·05	2222	2222	10·56	1174	1174	8·4	593	593
20	2	10·25	20·50	1077	2154	7·1	358	716	5·0	125	250
21	½	6·6	3·30	287	144	3·24	34	17	1·5	3	2
			446·35		134148			124701			114291

A.' W.L.

	Function of area.	*Mult.*	*Product.*	*Mult.*	*Moment.*
W.L.	446·4	5	2232	7	3125
W.L.	438·7	8	3510	6	2632
W.L.	424·5	−1	5742	−1	5757
			424		424
			5318		5333

Displacement of layer (L.W.L. to A.W.L.) $= 5318 \times \frac{4}{3} \times 19{\cdot}25 \times \frac{2}{12} \times \frac{1}{35} =$ 651 tons.

Moment about L.W.L. $= 5333 \times \frac{4}{3} \times 19{\cdot}25 \times \frac{4}{24} \times \frac{1}{35} =$ 652 ft.-tons.

Displacement to A.W.L. $= 3962$ tons.

C.B. below L.W.L. $= \frac{18210}{3962} = 4{\cdot}60'$

	Tons.	*Moment below* L.W.L.
Up to L.W.L.	3311	18862
ayer L.W.L. to A.W.L.	651	−652
	3962	18210

Tons per in. $= 446{\cdot}4 \times \frac{4}{3} \times \frac{19{\cdot}25}{420} = 27{\cdot}3$.

Transverse BM $= \frac{134148}{3962 \times 35} \times \frac{4}{3} \times \frac{19{\cdot}25}{3} =$ $8{\cdot}23'$.

2 W.L.

	Function of area.	*Mult.*	*Product.*	*Mult.*	*Moment.*
L.W.L	438·7	5	2194	3	1316
W.L.	424·5	8	3396	10	4245
3 W.L.	404·4	−1	5590	−1	5561
			4·4		404
			5186		5157

Displacement of layer (L.W.L. to 2 W.L.) = $5186 \times \frac{4}{3} \times 19{\cdot}25 \times \frac{2}{12} \times \frac{1}{35} =$ 634 tons.

Moment about L.W.L. = $5157 \times \frac{4}{3} \times 19{\cdot}25 \times \frac{4}{24} \times \frac{1}{35} =$ 680 ft.-tons.

	Tons.	*Moment about* L.W.L.
Up to L.W.L.	3311	18862
Layer L.W.L. to 2 W.L.	634	630
	2677	18232

Displacement to 2 W.L. = 2677 tons.

C.B. below L.W.L. $= \frac{18232}{2677} = 6{\cdot}82'.$

Tons per in. $= 424{\cdot}5 \times \frac{4}{3} \times \frac{19{\cdot}25}{420} = 25{\cdot}9.$

Transverse BM $= \frac{124701}{2677 \times 35} \times \frac{4}{3} \times \frac{19{\cdot}25}{3} =$ 11·4'.

3 W.L.

	Function of area.	*Mult.*	*Product.*	*Mult.*	*Moment.*
L.W.L.	438·7	1	438·7	0	—
2 W.L.	424·5	4	1698·0	1	1698·0
3 W.L.	404·4	1	404·4	2	808·8
			2541·1		2506·8

Displacement of layer (L.W.L. to 3 W.L.) = $2541 \times \frac{4}{9} \times \frac{19{\cdot}25 \times 2}{35} = 1242$ tons.

Moment about L.W.L. = $2506{\cdot}8 \times \frac{4}{9} \times \frac{19{\cdot}25 \times 4}{35} = 2452$ tons.

	Tons.	*Moment about* L.W.L.
Up to L.W.L.	3311	18862
Layer L.W.L. to 3 W.L.	1242	2452
	2069	16410

Displacement to 3 W.L. = 2069 tons.

C.B. below L.W.L. $= \frac{16410}{2069} = 7{\cdot}93'.$

Tons per in. $= 404{\cdot}4 \times \frac{4}{3} \times \frac{19{\cdot}25}{420} = 24{\cdot}7.$

Transverse BM $= \frac{114291}{2069 \times 35} \times \frac{4}{3} \times \frac{19{\cdot}25}{3} =$ 13·52'.

EXPLANATION OF DISPLACEMENT SHEET (see pp. 94, 95).

The length of the ship between perpendiculars is divided into twenty equal intervals, and the immersed depth by seven equally spaced water-planes, the lowest being 2 feet above the keel amidships. Below 7 W.L. is treated as an appendage, it being preferable in all cases not to take the lowest W.L. down to or very near to the keel. The ordinates or half-breadths at the intersections of the vertical cross sections with the horizontal sections are measured off in feet, and set down in dark figures (usually in red) in rows opposite their ordinate number and under their W.L. number. Water-lines that are snubbed or cut away at the ends should be produced to the perpendiculars by eye for the purpose of these measurements; the volumes thus added are afterwards deducted as negative appendages.

The Simpson's multipliers (halved in order to reduce the labour of multiplication) are placed against the ordinate and water-line numbers; each ordinate is multiplied by the multiplier appropriate to its ordinate number, the result being placed on the right; it is also multiplied by the multiplier appropriate to its water-line, and the result is placed underneath.

Adding the former products in columns gives the functions of the water-planes; these are multiplied by the appropriate water-line multipliers, and the products then added, giving a number (3231·0 in the text) which is a function of the displacement. The displacement

of the main solid is obtained by multiplying this function $\frac{2}{3} \times \frac{1}{35} \times$ spacing of W.L.'s × spacing of ordinates ; the factor $\frac{1}{6}$ is derived from Simpson's first rule applied twice in succession, the 8 is 2 for both sides of ship × 4 for the half multipliers used twice instead of the whole ones ; the $\frac{1}{35}$ converts cubic feet into tons (for sea-water).

The functions of the water-planes are again multiplied by the number of intervals from the L.W.L. ; the sum of the products (8706·78) being a function of the moment about the L.W.L. The multiplier $-\frac{2}{3} \times \frac{1}{35} \times$ (spacing of W.L.'s)^2 × spacing of ordinates — gives this moment in foot-tons.

Again, the products of the ordinates with the water-line multipliers are added in rows, the sums being functions of the transverse areas ; these are multiplied by the appropriate ordinate multipliers, and the products added, giving the same function of the displacement (3231·0) as before. These products (headed 'multiples of areas') are further multiplied by the number of ordinate spacings from amidships (station 11) ; the products are added for each end of the ship, and the difference between the sums gives a function (962·76) of the moment of the main solid about amidships. On using the multiplier $\frac{2}{3} \times \frac{1}{35} \times$ spacing of W.L.'s × (spacing of ordinates)^2, the actual moment is obtained in foot-tons.

The lower appendage is dealt with by calculating the half-area of each transverse section below the lowest W.L., and the vertical position of its c.g. They are tabulated on the left, as shown. Each semi-area is multiplied by its Simpson's multiplier, and the result by the number of intervals from amidships ; the functions of areas are also multiplied by the distances of their c.g. below 7 W.L. The three results are added in columns, allowance being made for the opposite signs of the two ends of the longitudinal moments, the sums are converted by the correct multipliers as shown. The remaining appendages are calculated by the ordinary rules for volumes and moments of solids, rough approximations being alone required. The 'recess' is that due to the emergence of the shafts. A table is set forth containing the displacements and moments of each item ; the total displacement and the position of the centre of buoyancy are then found by simple summation and division.

To obtain the position of the longitudinal metacentre (see p. 133), each ordinate of the load water-plane is twice multiplied by the number of intervals from amidships. The difference between the sums of the first products for each ends (326·52) is a function of the moment ; the multiplier $-\frac{1}{3} \times$ (longitudinal interval)^2 — gives the moment of the main portion about amidships. The sum of the second products multiplied by $\frac{1}{3} \times$ (longitudinal interval)^3 gives the moment of inertia of the main portion about amidships. Area of the main portion is obtained from the function of area (438·7) multiplied by the multiplier $\frac{1}{3} \times$ longitudinal interval. In this case the appendage aft, being fairly large, has an appreciable effect ; its area, moment, and moment of inertia are calculated (the last being equal to area × (distance of c.g. from amidships)^2) and inserted in a small table as shown. Thus the total area, the position of the centre of flotation (or c.g. of water-plane), and the moment of inertia about 11 are obtained. The correction necessary to find the inertia about the C.F. (see p. 70) is then introduced, and the longitudinal BM = moment of inertia about C.F. ÷ volume of displacement. By assuming an approximate vertical position for the c.g. of the ship, the moment to change trim 1 inch or $\frac{W \times GM\ (\text{long.})}{12\,L}$ is obtained. This, together with the position of the C.F., does not vary greatly with moderate changes of draft ; and they are generally assumed constant.

The remaining particulars evaluated are the tons per inch (equal to area of water-plane ÷ 420), the area of midship section (equal to function of area for 11 × $\frac{1}{3}$ × water-line interval), and the transverse BM. To obtain the last, the cubes of the ordinates of the water-plane are multiplied by Simpson's multipliers, and the products added. The

Length perpendiculars		385 ft. 0 in.
Breadth of hull		41 ,, 0 ,,
Draught of water	forward	13 ,, 0 ,,
	aft	14 ,, 0 ,,
	mean	13 ,, 6 ,,
Water-lines apart		42 ,, 0 ,,

Distance of sections from middle of length in terms of half length between perpendiculars.

I & IA	·0838
II & IIA	·3127
III & IIIA	·5
IV & IVA	·6873
V & VA	·9162

No. of Section.	Water-lines. 7 W.L. ½	6 W.L. 2	5 W.L. 1	4 W.L. 2	3 W.L. 1	2 W.L. 2	L.W.L. ½	Vertical Sections. Functions of Areas.	Differences.	Levers	Moments.
I	13·3	16·7	18·4	19·35	19·95	20·15	20·15				
	6·65	33·4	18·4	38·7	19·95	40·3	10·08	167·48			
IA	13·6	17·15	18·8	19·75	20·3	20·5	20·5				
	6·8	34·3	18·8	39·5	20·3	41·0	10·25	170·95	3·47	·0838	·296
II	10·2	13·55	15·6	16·7	17·4	17·8	18·0				
	5·1	27·7	15·6	33·4	17·4	35·6	9·0	143·8			
IIA	10·4	14·6	16·9	18·25	19·05	19·56	19·84				
	5·2	29·2	16·9	36·5	19·05	39·12	9·92	155·89	12·09	·3127	3·780
III	6·5	9·85	11·75	12·9	13·6	14·8	14·4				
	3·25	19·7	11·75	25·8	13·6	28·2	7·2	159·5			
IIIA	5·8	10·3	13·1	15·06	16·4	17·16	17·8				
	2·9	20·6	13·1	30·12	16·4	34·32	8·9	126·34	16·84	·5	8·42
IV	3·2	5·75	7·25	8·25	8·85	9·3	9·66				
	1·6	11·5	7·25	16·5	8·85	18·6	4·83	69·13			
IVA	2·3	4·65	7·3	9·85	11·95	13·45	14·54				
	1·15	9·3	7·3	19·7	11·95	26·9	7·27	83·57	14·44	·6873	9·925
V	0	1·5	3·25	2·55	2·65	2·7	2·76				
	0·	3·0	3·25	5·1	2·65	5·4	1·38	19·78			
VA	·76	1·0	1·2	2·5	4·4	6·6	8·4				
	·38	2·0	1·2	5·0	4·4	13·2	4·2	30·38	10·6	·9162	9·711
	66·06	95·35	112·55	125·16	134·55	141·32	146·06	1076·82			32·128
	33·03	190·7	112·55	250·32	134·55	282·64	73·03	1076·82			
	6	5	4	3	2	1	0				
	198·18	953·5	450·2	750·96	269·1	282·64	0	=2904·58			

Displacement (main solid) $=1076\cdot8\times2\times2\times2/3\times385/10\times1/35=3159$ tons.

Moment below L.W.L. $=2904\times6\times2\times2\times2\times2/3\times385/10\times1/35=17050$ ft.-tons.

Moment abaft $\mathbin{\rotatebox{90}{⋈}}=32\cdot13\times2\times2\times2/3\times385/10\times1/35\times385/2=18150$ ft.-tons.

No. of Section.	Tr. Metacentre. Ord. of Waterplane.	Cubes of Ords.	Centre of Flotation. Differences of Ordinates	Levers.	Moments.	Long. Metacentre. Sums of Ordinates	Squares of Levers.	Moments
I	20·15	8181						
IA	20·5	8615	·35	·0838	·029	40·65	·007	·285
II	18·0	5832						
IIA	19·84	7809	1·84	·3127	·575	37·84	·098	3·708
III	14·4	2986						
IIIA	17·8	5640	3·4	·5	1·7	32·2	·25	8·05
IV	9·66	902						
IVA	14·54	3074	4·88	·6873	3·354	24·2	·472	11·422
V	2·76	21						
VA	8·4	593	5·64	·9162	5·176	11·16	·840	9·374
		43653			10·835			32·859

Transverse BM $=\frac{I}{V}$

$=\frac{43653\times\frac{2}{3}\times38\cdot5}{3310\times35}$

$=9\cdot67$ ft.

Area L.W.L. $=146\cdot06\times2\times38\cdot5=11250$ sq. ft.

Long. BM $=\frac{94600000}{3310\times35}=317$ ft.

C.F. abaft $=\frac{10\cdot835}{146\cdot06}\times\frac{385}{2}=14\cdot28$ ft.

	A	h	Ah
Main area	11250	14·28	160800
Aft appendage	100	198	19800
Total area L.W.L.	11350		180600

C.F. of total abaft $\mathbin{\rotatebox{90}{⋈}}=\frac{180600}{11350}=15\cdot9$ ft.

I of main portion about $\mathbin{\rotatebox{90}{⋈}}=32\cdot84\times2\times38\cdot5\times\left(\frac{385}{2}\right)^2=93600000$ ft^4.

I of aft appendage $=100\times(198)^2=\underline{3900000}$ ft^4.

97500000

Deduct $Ah^2=11350\times(15\cdot9)^2=\underline{2900000}$

94600000

Displacement abaft $\mathbin{\rotatebox{90}{⋈}}=\frac{93600000}{3159\times35}=846$ tons.

Displacement forward $=3159-846=2313$ tons.

Displacement of forward portion about sheer by Simpson's Rules, giving a total displacement of 2310 tons.

sum multiplied by $\frac{3}{3}$ × longitudinal interval is equal to the moment of inertia of the water-plane about amidships; on dividing this by the volume of displacement, the transverse BM is obtained (see p. 111).

Frequently the displacement, tons per inch, transverse BM, and vertical position of the C.B. are required for other water-lines. Here they are worked out for 'A' W.L. (2 feet above L.W.L.), 2 W.L., and 3 W.L. The process consists of finding the volumes and moments of the layers between the respective W.L. and the L.W.L.; and adding to, or subtracting from, the displacement and moment for the L.W.L. The multipliers used at A.W.L. are 5, 8, −1 for volumes, and 7, 6, −1 for moments about middle ordinate (see pp. 46 and 68). For 2 W.L. the multipliers are 5, 8, −1 for volumes, and 3, 10, −1 for moments about end ordinate. For 3 W.L. the ordinary Simpson's multipliers are employed. When the after appendage is large, it is desirable to use the tons per inch and C.F., *both corrected for after appendage*, instead of the "functions of area" taken on p. 97. The transverse BM's are obtained by cubing the ordinates as for the L.W.L.

EXPLANATION OF DISPLACEMENT SHEET USING TCHEBYCHEFF'S RULE.

The horizontal water-lines are spaced equidistantly as with the preceding displacement sheet; an appendage is left below 7 W.L. The vertical transverse sections are spaced so as to meet the requirements of Tchebycheff's rule (see p. 46), using five ordinates for each half of the length, i.e. ten ordinates in all; the positions of the sections are indicated at the top of the sheet.* The ordinates are measured from the half-breadth plan; they are numbered I, II, III, IV, V for the fore end, commencing from amidships, and I A, II A, III A, IV A, V A for the after end, commencing from amidships. The half-breadths are measured off in feet and inserted in the table against the number of the corresponding ordinate, and under the corresponding water-line, in dark figures (usually in red). Under each water-line is set the correct Simpson's multiplier, halved for convenience; no multiplier is required opposite the ordinates.

The ordinates are first added in columns, the sums being functions of areas of the water-planes. These are multiplied by the corresponding Simpson's multipliers and their sum (1076·82) is a function of the displacement; the multiplier required to obtain the displacement in tons is 2 (for both sides) × $\frac{3}{3}$ (Simpson's rule for half multipliers) × $\frac{1}{35}$ (salt water) × water-line spacing $\frac{\times \text{length}}{10}$. The products of the functions of the water-line areas are also multiplied by the number of intervals from the L.W.L.; the results are functions of vertical moments. The sum of these multiplied by $\frac{3}{3}$ × $\frac{1}{35}$ × (water-line spacing)^2 × $\frac{\text{length}}{10}$ is the moment of the main solid about the L.W.L.

Each ordinate is afterwards multiplied by its appropriate water-line multiplier, and the products added in rows; the sums are functions of the areas of the transverse sections. The sum of these gives a function of the displacement, which should be the same as that previously obtained (1076·82). The differences between the functions of the areas for the fore and after ends, taken in pairs, are written down and multiplied by the levers equivalent to the Tchebycheff spacings expressed in terms of half the length; in the example the functions for the after body are greater in each case than the corresponding ones for the fore body, but, if this is not the case, allowance should be made for sign. The products are functions of moments about amidships; their sum multiplied by $\frac{3}{3}$ × $\frac{1}{35}$ × water-line spacing × $\frac{(\text{length})^2}{20}$ is the moment of the main solid about amidships expressed in foot-tons.

Unless a special body has been constructed with Tchebycheff sections, the calculation of the lower appendage is the same as that in the ordinary displacement sheet, equidistant sections being used.

* Alternatively the ordinary 'Simpson' sections numbered 2, 5, 7, 10, 12, 15, 17 and 20, may be taken instead of the exact sections required for Tchebycheff's rule with four ordinates repeated.

The remaining appendages are calculated as before; the final table has not been inserted in this case.

The calculation for the transverse metacentre is the same as before except that the cubes of the ordinates are added direct, and their sum multiplied by $\frac{2}{3} \times \frac{\text{length}}{10}$ to obtain the transverse moment of inertia.

For the centre of flotation, the differences of the ordinates in pairs are written down, allowing for sign if necessary, and multiplied by the Tchebycheff levers. The sum of the products is a function of moments; and the distance of the C.F. abaft amidships is equal to $\frac{\text{function of moments}}{\text{function of areas}} \times \frac{\text{length}}{2}$. The area of the water-plane is equal to function of area $\times 2 \times \frac{\text{length}}{10}$. To obtain the longitudinal moment of inertia, the ordinates are added in pairs, and multiplied by the squares of the Tchebycheff levers (·007, ·038, ·25, ·472, ·840). The sum of the products multiplied by a twentieth of the cube of the length is the moment of inertia about amidships. The corrections for the after appendage, and for the position of the C.F., and the calculation of BM are similar to those on the ordinary sheet.

The remaining calculations are made as before.

WEIGHT AND CENTRE OF GRAVITY OF SHIPS.

In the early stages of design, an approximation must be made to the weight of a ship in order to determine whether it is equal to the displacement assumed; the position of the centre of gravity is also required in order to determine the stability and trim.

The weight of a ship is conveniently divided into six items: hull, equipment, machinery, fuel, armour, and armament. In a merchant vessel the two last named are replaced by the load to be carried. The proportions vary greatly in different ships; those in the table are illustrative of certain types:—

Type of Ship.	Percentage Weight.					
	Hull.	Equipment.	Machinery.	Fuel (normal).	Armour.	Armament or Load.
Battleship . .	34	3	10	3½	31½	18
First-class Cruiser	38	4	17	7	20	14
Light Cruiser. .	43½	6	20	12	12½	6
T. B. Destroyer (deep condition)	34	4	34	25	—	3
Steam Yacht . .	61	10	19	10	—	—
Atlantic Liner .	55	included in hull.	27	16	—	2 (Passengers and Stores)

Hull.

First Estimate.

The weight of hull is determined to a first approximation * in a variety of ways. In ships of very similar type it may be assumed to be the same percentage of the displacement, e.g. 34% in battleships, etc. Or it may be compared with the product length × (breadth + depth) amidships, the coefficient being determined from a similar ship, making allowance for any great alteration of scantlings. Mr. J. Johnson (in Trans. Inst. Nav. Archs., 1897) published a useful method for approximating to the hull weight of a vessel built to the highest class at Lloyd's or Veritas.

If N = a modification of Lloyd's *old* longitudinal number
= Length from after part of stem to fore part of stern post on upper deck beams × {⅓ greatest moulded breadth + depth from top keel to top upper deck beams + ½ midships girth to upper deck stringer}.

In spar- and awning-deck vessels the girths and depths are measured to the spar or awning decks ; they are taken to the main deck in one-, two-, and well-decked vessels.

W = finished weight in tons of the steel hull.

$$\text{Then } W = K\left(\frac{N}{100}\right)^{\chi}$$

$$\text{or } \log_{10} W = \chi \log_{10}\left(\frac{N}{100}\right) - \lambda.$$

Where χ and K or λ are determined from the table below :—

Type of Vessel.	χ	K	λ
Three deck	1·40	0·492	0·308
Spar deck	1·35	0·576	0·240
Awning deck	1·30	0·665	0·177
One-, two-, and well-deck . . .	1·30	0·856	0·068
Sailing	1·40	0·410	0·387

The distance of G abaft the middle of length and above the keel can be estimated from information available for other ships, taking these distances proportional respectively to the length and total depth.

Detailed Estimate.

In the later stages, when scantlings are fixed, the weight and centre of gravity of hull are found as follows:—

The hull may be divided into two groups : (1) calculable items forming about 60% (in a warship) of the whole, which include the greater part of the structure, and (2) 'judgment' items, including portions of complicated structure and fittings.

The latter can only be assessed by comparison with known weights in a similar ship (recorded weights if possible) ; the centre of gravity of each item can usually be determined

* See also under "Design".

with fair accuracy from its position. The former are directly calculated from the scantlings ; the manner of so doing is indicated in a few instances below. To each item 3% should be added for fastenings (or such an addition should be made at the end of the calculation).

If the stresses on the ship are also required, it is convenient to divide every item into portions wholly before and wholly abaft the midship section. Moments are taken about two fixed planes, one being generally the midship section and the other the L.W.L. or the keel.

Outer Bottom Plating.—Assume all of uniform thickness, repeating as necessary for the portions where the thickness is in excess or in defect of that assumed. Divide the length into sections spaced equidistantly, and measure the half-girths at each section. Apply the method of Rule II, par. 24, p. 58, obtaining the 'modifying factor' at each section. (This is the ratio of the slant length of the mean water-line intercepted between the sections to their perpendicular spacing.) Find the height of the centre of gravity of each section by dividing it into four equal parts, and proceeding as in par. 16, p. 62. Then arrange the calculations as in the following table for the forward portion of a warship below the armour deck, the percentage for laps, butts, and liners being taken as calculated for an average-sized plate (say $20' \times 4'$).

Transverse Framing.—There are usually several varieties, such as web and ordinary frames, or bracket, lightened plate, and watertight frames. To avoid calculating each one separately, calculate the weight and height of c.g. for a specimen frame at intervals of about $\frac{1}{10}$ length. Plot these as curves on a base of length of ship, drawing separate curves for each type of frame. The weight and c.g. position can then be read off for each frame, or a mean can be taken for a group of similar frames coming together.

Longitudinal Framing.—Usually uniform in section, but the height of c.g. must be taken at equidistant intervals, and the mean taken.

Bulkheads.—Main bulkheads are usually thicker towards the bottom, and the c.g. is below the centre of area. Take the minimum thickness of plating and calculate its weight and c.g. without any allowance. Then add the additional thickness, the stiffeners, and allowances for laps, butts, and fastenings ; all but the first item have the c.g. at centre of area. Then find the ratio of the initial to final weights, and of the c.g. below centre of area to the height of bulkhead. If this be done for one or two typical bulkheads, the rest may be determined from the simpler first calculation by allowing the same ratios. The ratio of weight for the main bulkheads of a warship is 1 : 1·9 ; for ordinary below-water bulkheads it is 1 : 1·66.

Weight and Centre of Gravity of Part of Outer Bottom Plating.

No. of Station.	Half-girth.	Modifying Factor.	Product.	Simpson's Multiplier (halved).	Function for Area.	Multiplier for Longitudinal Moment.	Product.	Depth c.g. of girth below L.W.L.	Product. *
1	21·0	1·03	21·6	½	10·8	5	54·0	·6	6·5
2	27·2	1·03	28·0	2	56·0	4	224·0	1·2	67·2
3	30·8	1·03	31·7	1	31·7	3	95·1	1·8	57·0
4	34·6	1·02	35·3	2	70·6	2	141·2	2·1	148·0
5	38·8	1·02	39·6	1	39·6	1	39·6	2·2	87·2
6	41·5	1·01	41·9	2	83·8	0	553·9	2·3	192·7
7	42·6	1	42·6	1	42·6	1	42·6	2·3	98·0
8	43·7	1	43·7	2	87·4	2	174·8	2·4	209·6
9	43·9	1	43·9	1	43·9	3	131·7	2·4	105·3
10	44·0	1	44·0	2	88·0	4	352·0	2·4	211·2
11	44·0	1	44·0	½	22·0	5	110·0	2·4	52·8
Sum.	Common interval 20 feet.				576·4		811·1		1235·5

Area (both sides) = $576{\cdot}4 \times \frac{4}{3} \times 20 = 15{,}370$ sq. ft. Percentage for laps, etc., assumed 22; for fastenings 3. Plating $\frac{1}{2}''$ thick.

Weight = $15{,}370 \times 20{\cdot}4 \times 1{\cdot}22 \times 1{\cdot}03/2240 = 176$ tons.

c.g. abaft No. 6 = $\frac{811{\cdot}1}{576{\cdot}4} \times 20 = 28{\cdot}2'$; i.e. c.g. is $100' - 28{\cdot}2'$ or $71{\cdot}8'$ before amidships (No. 11).

c.g. below L.W.L. = $\frac{1235{\cdot}5}{576{\cdot}4} = 2{\cdot}15'$.

* This column is the product of the preceding by the one headed 'Function for Area'.

For smaller bulkheads between decks, measure total length and multiply by mean distance between decks. From this area the weight of plating is at once found ; 12½% is a usual percentage to add for laps. The length of the boundary bars is readily determined ; that of the stiffeners is found by dividing the area by their spacing, adding a percentage as necessary for brackets at heads and heels.

Decks.—Take in sections, each section having uniform thickness. Find area and longitudinal c.g. of each portion. Weight of plating is found as with bulkheads ; that of beams is equal to area × weight of beam per foot run ÷ beam spacing, adding a small allowance for beam knees or brackets. For planking, find volume of wood and multiply by its density, allowing about 6% for fastenings, e.g. a 3 in. deck weighs 14 lb. per square foot if of teak, 12 lb. if of fir, including fastenings.

When all the items for the hull have been evaluated, they are tabulated in the manner shown in the succeeding pages. The calculations therein given are those for a paddle tug 145′ × 28′ × 11′ 4″ (4 feet freeboard) × 750 tons displacement.

SUMMARY OF WEIGHT OF HULL.

Note.—Base for vertical C.G.s top of bar keel.

Item.	Tons.	Before 6.		Abaft 6.		Above base.	
		C.G	Mmt.	C.G	Mmt.	C.G.	Mmt.
Outer bottom plating . .	64·5	—	—	2·0	129·0	6·7	432·0
Bar keel	3·5	—	—	—	—	·33	1·2
Stem	·4	72	28·8	—	—	10·5	4·2
Sternpost	·4	—	—	72·5	29·0	8·6	3·4
Rudder, etc.	1·7	—	—	74·0	125·8	11·0	18·7
Intercostal M.L. keelson	2·5	—	—	—	—	·9	2·3
Transverse bulkheads . .	15·0	·5	7·5	—	—	9·0	135·0
Transverse framing (ex webs)	27·0	—	—	3·8	102·6	5·0	135·0
Cant frames	·6	—	—	75	45·0	16·0	9·6
Deep beams and web frames	10·0	—	—	1·1	11·0	9·5	95·0
Side keelsons and bilge stringers	8·5	—	—	3·0	25·5	3·4	28·9
Top of reserve feed tank	2·3	18	41·4	—	—	2·0	4·6
Upper deck	38·0	—	—	3 5	133·0	16·0	608·0
Lower deck forward . .	5·0	44	220·0	—	—	8·7	43·5
Lower deck aft . . .	4·5	—	—	52	231·0	8·5	38·2
Bridge deck	10·0	6·6	66·0	—	—	22·6	226·0
Sponsons, etc.	12·5	2·5	31·2	—	—	13·5	168·8
Paddle-boxes	12·0	2·5	30·0	—	—	19·0	228·0
Coal-bunker bulkheads .	7·3	—	—	19	138·7	7·9	57·7
Division in chain locker	·3	55	16·5	—	—	4·0	1·2
Divisions in fresh-water and ballast tanks . .	·6	—	—	—	21·2	—	3·2
Topsides	4·5	—	—	6	27·0	18·1	81·5
Main engine bearers . .	14·5	14	203·0	—	—	4·5	65·2
Auxiliary engine bearers	3·0	3	9·0	—	—	1·8	5·4
Boiler bearers	1·5	—	—	19·1	28·6	2·1	3·1
Carried forward . .	250·1		653·4		1050·4		2397·3

Item.	Tons.	Before 6.		Abaft 6.		Above base.	
		C.G.	Mmt.	C.G.	Mmt.	C.G.	Mmt.
Brought forward.	250·1		653·4		1050·4		2397·3
Houses and fittings on and above bridge	3·5	15·5	54·2	—	—	27·0	94·5
Sponson houses	4·5	2·5	11·2	—	—	18·5	83 2
Rubber	10·0	—	—	2	20·0	15·0	150·0
Companions	2·5	—	—	2	5·0	18·7	46·8
Skylights	2·0	—	—	·5	1·0	21·5	43·0
Engine and boiler casings	9·0	—	—	11·8	106·2	19·0	171·0
Steering gear	·1·5	—	—	22 0	33·0	16 5	24·8
Ventilation	1·2	—	—	5 0	6·0	18 0	21·6
Pumping	1·5	8	12·0	—	—	10·0	15·0
Pillars	·5	—	—	5·0	2·5	12 0	6·0
Cathead	·4	61	24·4	—	—	21·0	8·4
Mooring pipes and chocks	3·2	—	—	—	—	19·0	60·8
Horn	·4	73	29·2	—	—	22·0	8·8
Towing hooks and stiffening	1·1	—	—	25·0	27·5	19·0	20·9
Towing beams and supports	·9	—	—	57 0	51·3	20·5	18·5
Bollards and fairleads	12·0	—	—	7·5	90·0	16·5	198·0
Samson post	·5	64	32·0	—	—	18·0	9·0
Boats davits	2·2	—	—	22	48·4	22·0	48·4
Awning stanchions, etc.	·8	—	—	—	—	22·5	18·0
Ladders	1·0	—	—	—	—	15 0	15·0
Fittings in galley	1·2	—	—	13 0	15·6	17·0	20·4
W.C. fittings	·5	2·5	1·2	—	—	17·0	8·5
Fittings in other sponson houses	·3	16	4·8	—	—	17·0	5·1
Miscellaneous upper deck fittings	6·0	—	—	3·5	21·0	16·5	99·0
Shovelling flat	1·2	—	—	18·3	21·9	1·5	1·8
Side scuttles forward and aft	1·0	2 0	2 0	—	—	14·0	14·0
Side scuttles to sponson houses	·3	—	—	—	—	20·0	6·0
Coaling scuttles	·6	—	—	18	10·8	15·3	9·2
Engineers store-room bulkhead	1·3	1·5	1·9	—	—	6·5	8·5
Cabin bulkheads forward	·5	33	16·5	—	—	12	6·0
Cabin bulkheads aft	·9	—	—	48	43·2	12	10·8
Corticene on lower deck aft	·4	—	—	50	20·0	8·5	3·4
Silicate lagging	·3	—	—	36·7	11·0	9	2·7
Chain locker fittings	·5	55	27·5	—	—	3	1·5
Fittings in—							
Fore crew space	1·0	55	55·0	—	—	12	12·0
After crew space	2·0	38	76·0	—	—	12	24·0
Cabins forward	·6	32	19·2	—	—	12	7·2
Cabins and mess aft	1·5	—	—	44	(6 0	12	18·0
Cabin and saloon	1·5	—	—	48	87·0	12	18·0
Fore holds	1·5	45	67·5	—	—	5·5	8·3
After holds	1·5	—	—	48	72·0	5·5	8·3
Engineers' Stores	·5	—	—	—	—	7·0	3·5
Topside fittings	1·5	—	—	—	—	18·0	27·0
Cement	8·0	—	—	3	24·0	1·1	8·8
Paint	4·0	—	—	—	—	10	40 0
	347·4		10·8·0	2·14	1831·3 10 8 0 746·3	11 0	3831·0

C.G. abaft 6, 2·14′. Above base, 11′.
Take 350 tons. 2·1′ abaft 6. 11′ above base.

Equipment.

This is conveniently divided (in warships) as follows: Fresh water (for 10 days allow about 8·7 tons for 100 men); provisions and spirits, including tare (for 4 weeks allow about 5·7 tons per 100 men); officers' stores and slops; officers, men, and effects (allow 8 to the ton); masts, rigging, sails, etc.; cables (500 fathoms); anchors; boats; warrant officers' stores (4 months); torpedo net defence.

In passenger ships undergoing long voyages allow 1 ton per 5 persons for passengers' gear, including baggage, bedding, etc., also ·025 ton per day per person (average) for water and provisions.

Machinery and Fuel.

In the preliminary estimate the weight of machinery is based on the total power. Coefficients for various types of machinery are given on pp. 389, 390. Information obtained from actual ships should also be utilized where possible.

The weight of coal assumed is frequently an arbitrary amount less than the full bunker capacity. The full coal storage can be, however, determined from the volume of the bunkers calculated by the rules on p. 54, the areas of the sections being taken to underside of beams only. About 10 or 15 per cent (varying with type and shape of bunker) is then deducted for broken stowage; the net volume in cubic feet, on being divided by 43 (North Country coal), 40 (Welsh coal), 36 (patent fuel symmetrically stowed) or 45 (patent fuel shot into bunkers), gives the stowage in tons.

The weight of liquid fuel is equal to the whole volume in cubic feet divided by 38·5.

In all cases the centre of gravity of the fuel is the c.g. of the volume (see p. 66).

Armour.

The weight and position of c.g. of armour in warships are determined by a process similar to that adopted for the hull. If the armour is not specified by its weight per square foot of plate, this can be determined from its thickness, since it weighs 495 lb. per cubic foot. Add 1½% for bolts. Backing is dealt with similarly to the planking of a teak deck.

Armament or Load.

The weight of guns, mountings, charges, and projectiles are known (see pp. 380-385), and the position of the c.g. of each item can generally be spotted without difficulty. Allow 30 to 40% tare for cartridge cases.

The load in a cargo ship is generally determined beforehand. Its c.g. is usually found by assuming the whole space available to be filled with a homogeneous cargo—the assumption the most unfavourable to the stability; the c.g. is then that of the volume of the hold.

For passengers without baggage allow 16 to the ton with men, women, and children; 14 to the ton with men only. In pleasure steamers where the stability can be affected, assume the c.g. of passengers seated to be 6 inches above the seat; for those standing, 2 feet above the deck is generally a safe assumption.

The final weight and position of centre of gravity are found by adding together the weights and moments of the several portions as shown in the table below :—

SUMMARY OF BATTLESHIP WEIGHTS (580′ × 90′ × 27′ 6″ – 44′ deep).

Item.	Weight.	Moment from amidships.	Moment above L.W.L.	Moment below L.W.L.
Weight before amidships—	Tons.	Ft.-tons.	Ft.-tons.	Ft.-tons.
Armament	2076	236110	43020	1870
Armour	3270	353300	42077	1901
Hull	3770	355340	25473	28796
Machinery	820	29500	—	8200
Coal	670	23000	1360	5840
Equipment	400	55000	7420	766
		1052250		
Weight abaft amidships—				
Armament	2763	348020	46280	2190
Armour	3585	481250	28848	3563
Hull	4933	597170	24849	38897
Machinery	1930	198800	—	19300
Coal	330	21400	640	3060
Equipment	335	48200	3564	52
	24882	1694840	223531	114435
		1052250	114435	
		642590	109096	

Total weight of ship = 24882 tons.

c.g. abaft $\mathfrak{X} = \frac{642590}{24882} = 25{\cdot}7$ feet.

c.g. above L.W.L. $= \frac{109096}{24882} = 4{\cdot}38$ feet.

STABILITY.

If a ship be slightly disturbed from a position of equilibrium, and if the forces then in operation tend to restore the original position, the equilibrium is termed *stable* ; if the forces tend to move it further from the original position, the equilibrium is termed *unstable* ; if it shows no tendency to move away from or return to the original position, the equilibrium is termed *neutral*.

The equilibrium of a ship is always stable as regards vertical deflections causing an alteration of displacement ; the only disturbances that need examination consist of inclinations about horizontal axes with the displacement unaltered. Of these the principal are : (*a*) inclination in a transverse plane about a longitudinal axis, and (*b*) inclination in a longitudinal plane about a transverse axis. The stability in these directions is termed transverse and longitudinal respectively.

Transverse Stability.

Fig. 119 is a transverse section of a ship heeled over through a certain angle θ. W'L' is the water-line for the inclined position, and WL is the water-line for the upright position. These two planes intersect each other in a longitudinal direction, and bound two wedges L'SL and WSW' equal in volume to each other, provided the displacement remains the same. The wedges are called respectively the wedges of *immersion* and *emersion*, or the *in* and *out* wedges. G is the centre of gravity of the ship and B' her centre of gravity of displacement, or centre of buoyancy. The weight of the ship then acts vertically downwards through G, and the resultant pressure of the water acts vertically upwards through B', these two forces forming a *righting couple*, the arm of which is GZ—that is, the perpendicular distance between the lines of action of the two forces. The moment of this couple—that is, the weight of the ship, or its displacement, multiplied by the length of the arm GZ—is the *moment of statical stability* of the ship at the given angle of inclination θ. This moment is generally expressed in *foot-tons*—that is, the weight of the ship in tons multiplied by the length of the arm GZ in feet. B is the centre of buoyancy of the ship when upright ; S is the point of intersection of the two water-lines, I the point where the vertical B'M cuts the plane of flotation ; g and g'

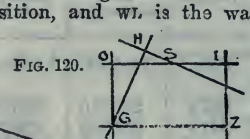


Fig. 120.

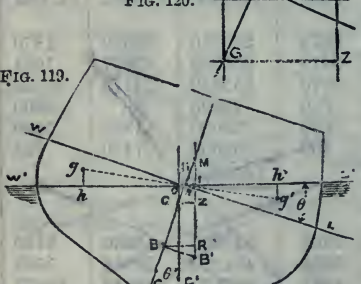


Fig. 119.

are the centres of gravity of the emerged and immersed wedges respectively, gh and $g'h'$ being perpendiculars dropped to g and g' from the plane of flotation W'L'. The point M, where the vertical line BM, drawn through the centre of buoyancy B when the ship is in an upright position, cuts the vertical line B'M, drawn through the centre of buoyancy B' for the inclined position, is termed the *transverse metacentre* when the ship is inclined through an indefinitely small angle, and also when the point of intersection is the same for all angles of heel.

If the centre of gravity G is below the metacentre M, the equilibrium is stable; if G is above M, the vessel is unstable, and will capsize or at least heel to a large angle; if G coincides with M, the equilibrium is neutral.

The intersection of the new vertical through B' is found usually to pass very near the metacentre M for all angles of heel up to 10° or 15°. Within these limits the stability lever GZ is equal to GM . sin θ; or the moment of statical stability is W . GM sin θ.

For moderate angles the stability depends wholly on the value of GM, which is termed the metacentric height. The position of G is calculated by the rules given on pp. 102-9; that of M is obtained by the process indicated below.

To obtain the height of the transverse metacentre.

Assume the angle of heel θ to be small, let

y = half breadth WS or SL at any station.

v = volume of either wedge WSW' or LSL'.

g, g_1 = c.g.s. of the wedges.

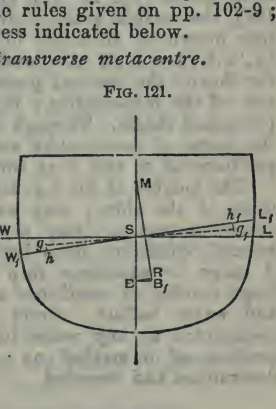
h, h_1 = feet of perpendiculars from g, g_1 on WSL'.

V = volume of displacement.

I = moment of inertia of water-plane about longitudinal axis through S.

dx = an element of length of ship.

FIG. 121.



Then $\text{BR} = \frac{v}{\text{V}} \cdot hh_1$.

Also $v \cdot hh_1$ = moment of transference of wedges.

$$= 2\int \frac{y^2\theta}{2} \cdot \frac{2y}{3} \cdot dx \text{ approximately.}$$

$$= \tfrac{2}{3}\theta \int y^3\, dx = \theta \cdot \text{I}.$$

$$\therefore \text{BR} = \theta \cdot \text{I}/\text{V}.$$

Also BR = θ . BM approximately.

Whence $\text{BM} = \frac{\text{I}}{\text{V}}$

The height of the transverse metacentre above the centre of buoyancy is equal to the moment of inertia of the water-plane area about the axis of inclination (in this case the centre line) divided by the volume of displacement.

The actual calculations for the transverse BM at several draughts of a ship are given in the displacement sheets on pp. 94, 100. The moment of inertia I is there expressed by the integral $\frac{2}{3}\int y^3 . dx$; the cubes of the ordinates are first integrated, and the result muliplied by the factor $\frac{2}{3}$.

In a ship whose sections are circular in the neighbourhood of the water-line, such as a submarine, the metacentre is coincident with the centre of the circular arcs.

Definition.—The *surface* stability of a ship is that obtained when the centre of gravity coincides with the upright centre of buoyancy.

Metacentric Diagram.

The stability of a ship in various conditions is conveniently exhibited by means of a metacentric diagram. In fig. 122, which shows the diagram for the ship taken in the displacement sheets (pp. 95, 97), and inclining experiment (pp. 135, 138), the vertical scale of draught is intersected by a straight line drawn at an angle of 45°. From the intersection of 'A.'W.L., L.W.L., 2 W.L., and 3 W.L., with this line are set up (or down) the vertical positions of the centres of buoyancy, and of the metacentres; these being obtained from the displacement sheets. Through the spots thus obtained are drawn the *curve of metacentres* and *curve of buoyancy*, giving the positions of M and B at intermediate water-lines.

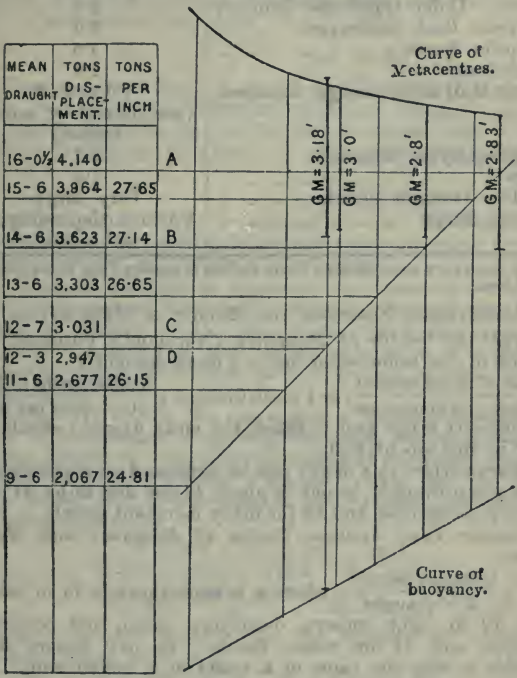
The heights of the c.g. are calculated for a number of conditions of the ship; they are here shown for the inclining condition (see *inclining experiment*, p. 135), the legend (or normal) condition, deep load condition, and light condition. A cargo or passenger ship is frequently worked out for a large number of conditions as regards stowage of cargo, coal, and water ballast. From the curves the position of the metacentre at any water-line is obtained, and the vertical position of G marked on; the metacentric height is thus determined and recorded.

Value of GM in Typical Ships.

A vessel having a low metacentric height is termed *crank*; one provided with a large GM is termed *stiff*. A crank vessel usually rolls less and moves more easily among waves than a stiff vessel; for this reason the value of the GM adopted for a ship is made as small as possible, consistent with safety and other considerations. Typical values are given in the following table :—

FIG. 122.

METACENTRIC DIAGRAM OF SMALL CRUISER.



A. Deep condition. Coal 725, oil 142, reserve feed 92 tons.
B. Normal condition. Coal 450 tons.
C. Light condition. No coal; no consumable stores.
D. Condition as inclined.

Note.—The curve of buoyancy is generally nearly straight; the tangent of its inclination to the horizontal is equal to $12 \times$ depth of C.B. below W.L. $\times \frac{\text{Tons per inch}}{\text{Tons displacement}}$

Type of Ship.	Minimum GM in feet.
First-class Battleships—Modern .	5·0
Do. Older types and Cruisers	3·5
Torpedo Boat Destroyers . . .	2·0
Torpedo Boats	1·5
Steamboats	1·0
Large Mail and Passenger Steamers	1·0 to 2·0* (maintained by water ballast)
Cargo-carrying Steamers . . .	2·0
Tugs	1·5
Shallow Draught Vessels	Very large
Sailing Ships	1·5 to 6 (depending on sail area)

* In some very large modern liners the GM is greater; e.g. in *Aquitania* GM is 4 feet.

Approximate Formulæ for Height of Metacentre.

Depth of Centre of Buoyancy (Normand's Formula).

Depth of C.B. below water-line $= \frac{1}{3}$ mean draught + $\frac{\text{volume of displacement}}{3 \times \text{area of water-plane}}$; or $\frac{1}{3}$ mean draught + $\frac{\text{displacement in tons}}{36 \times \text{tons per inch}}$

Note.—If a bar keel is fitted, the mean draught should be taken to the top of keel.

Alternatively, this depth can be expressed as a percentage of the mean draught, which is about 42 for fine ships, 44 for ordinary battleships, and 46 for many merchant vessels.

Distance (BM) *between Centre of Buoyancy and Metacentre.*

$\text{BM} = \frac{(\text{greatest beam})^2}{\text{K} \times \text{draught}}$; where K is approximately 13 in battleships, 12 in light cruisers, destroyers, cargo, and passenger steamers, and 11 in steam yachts. In new designs it is advisable to take the value of K found in a similar ship.

Stability at Large Angles of Heel.

If the heel be so large that the vertical through B′ (fig. 119) no longer intersects the middle line at a fixed point, the metacentric method is no more applicable.

During the inclination of the ship the centre of buoyancy moves from B to B′, and B′ lies in a plane parallel to a line joining g and g'. The distance BB′ can be found from the following expression :—

$$\text{BB}' = \frac{v \times gg'}{\text{V}}$$

where V = volume of displacement and v = volume of either of the wedges ;

$$\text{BR} = \frac{v \times hh'}{\text{V}}, \text{ where BR is perpendicular to B'M;}$$

and $$\text{GZ} = \text{BR} - \text{BG} \,.\, \sin\theta = \frac{v \times hh'}{\text{V}} - \text{BG} \,.\, \sin\theta,$$

whence Atwood's formula for expressing the *moment of statical stability* at any angle θ is

$$\text{M} = \text{W}\left\{\frac{(v \times hh')}{\text{V}} - (\text{BG} \,.\, \sin\theta)\right\}$$

The moment of *statical surface stability* at any angle θ is BR × W, being the righting moment obtained on the assumption that the c.g. of the ship coincides with B. The angle of heel in fig. 119 is BMB′ = LSL′, and its sine is equal to $\frac{\text{BR}}{\text{BM}} = \frac{\text{GZ}}{\text{GM}}$

Dynamical stability is defined to be the amount of mechanical work necessary to cause a body to deviate from its upright position, or position of equilibrium.

Dynamical stability is expressed as a moment by multiplying the sum of the vertical distances through which the centre of gravity of the ship ascends and the centre of buoyancy descends (i.e. the vertical separation of G and B), in moving from the upright to the inclined position, by the displacement.

In fig. 119 during the inclination of the ship through the angle θ, the centre of gravity has been moved through a vertical height GH − GO, and the centre of buoyancy has been lowered through a vertical distance B′I − BH, and the whole work to do this, or her moment of dynamical stability for the given angle θ, is

$$= \text{W}\{(\text{GH} - \text{GO}) + (\text{R}'\text{I} - \text{BH})\}$$
$$= \text{W}(\text{B}'\text{Z} - \text{BG}) = \text{W}(\text{B}'\text{R} - \text{BG} \,.\, \text{vers}\,\theta)$$
$$= \text{W}\left(\frac{wv\,(gh + g'h')}{\text{W}} - \text{BG} \,.\, \text{vers}\,\theta\right);$$

whence Moseley's formula for the moment of dynamical stability at any angle θ is

$$= wv(gh + g'h') - (\text{W} \times \text{BG} \,.\, \text{vers}\,\theta),$$

where W is the density of water.

The dynamical stability of a ship at any angle θ is the *integral* of its statical stability at the given angle—that is, if M = the statical stability and U the dynamical stability, then

$$\text{U} = \int \text{M}\,d\theta,$$

where $d\theta$ is a very small angle of heel.

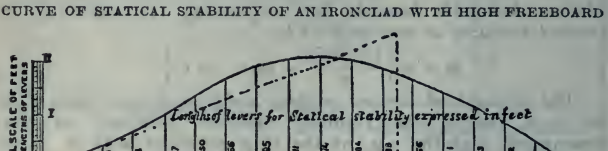
The *moment of dynamical surface stability* is expressed by multiplying the weight of the ship, or displacement, by the depression of the centre of buoyancy during the inclination—that is, for the angle θ

$$\text{U} = \text{W}(\text{B}'\text{I} - \text{BH}).$$

The Curve of Statical Stability is a curve used to record the value of the stability lever (GZ) of a vessel at any given angle of heel.

FIG. 123.

CURVE OF STATICAL STABILITY OF AN IRONCLAD WITH HIGH FREEBOARD


DECIMAL SCALE OF FEET FOR LENGTHS OF LEVERS
Lengths of levers for Statical Stability expressed in feet
SCALE OF DEGREES FOR ANGLE OF HEEL

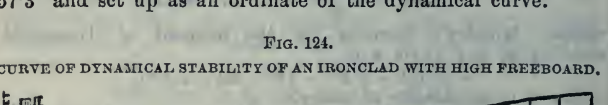
Method of Construction.—Calculate the length of the arm of the righting couple, or GZ (see fig. 119), for several successive angles of heel taken between the upright position and that at which the length of the arm becomes zero ; set the lengths of these arms off as ordinates (see fig. 123) from a base line the abscissæ of which represent to scale the respective angles of heel : a curve bent through the extremities of these ordinates will form a curve of statical stability.

Note.—The length of the perpendicular at 57·3° (one radian) intercepted between the tangent at the initial portion of the curve and the base line is equal to the metacentric height.

The Curve of Dynamical Stability is constructed in a similar manner to that of the curve of statical stability, with the exception that the various lengths of the arm (B′Z − BG) = (B′R − BG vers θ) (see fig. 119) are taken as ordinates instead of GZ. Or preferably the curve is obtained by integrating the statical curve. The area up to each ordinate of the statical curve expressed in degrees × feet is divided by 57·3° and set up as an ordinate of the dynamical curve.

FIG. 124.

CURVE OF DYNAMICAL STABILITY OF AN IRONCLAD WITH HIGH FREEBOARD.

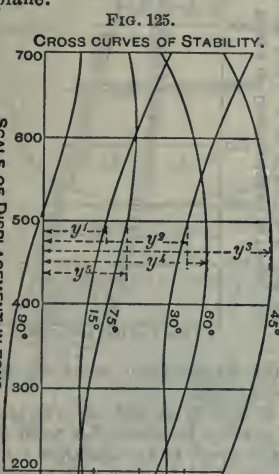

DECIMAL SCALE OF FEET FOR LENGTHS OF LEVERS
Lengths of levers for Dynamical Stability expressed in feet
SCALE OF DEGREES FOR ANGLE OF HEEL

Note.—The angle at which the statical lever vanishes (and at which the dynamical lever is a maximum) is termed the *range* of stability.

Cross Curves of Stability.

These curves may be termed 'vertical curves of stability'; they consist of curves of righting levers at various draughts or displacements for certain fixed angles of heel. They hold a somewhat similar relation to the ordinary curves of stability as the body plan of a ship does to its water plane.

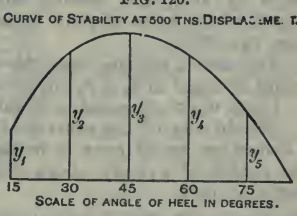
Fig. 125.



For cross curves (see fig. 125) the righting levers are calculated at certain fixed degrees of heel at various displacements, and the levers are set up as ordinates from an axis the abscissæ of which represent the displacement at which the levers for the fixed degree of heel are found.

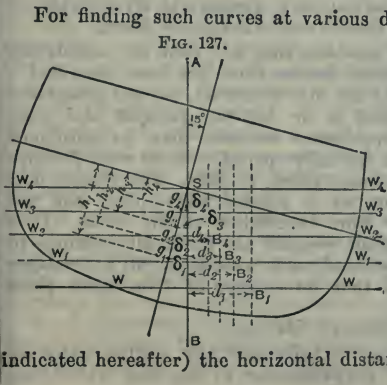
A number of such curves are constructed for various inclinations, and set off as in fig. 125.

Fig. 126.



For finding such curves at various draughts and angles of heel, say at 15° (see fig. 127), divide the body plan by a number of parallel planes representing various draughts of water or displacements.

Fig. 127.



Drop a perpendicular through the point where the highest water-line cuts the middle line of the ship, and then calculate (by the methods indicated hereafter) the horizontal distances d_1, d_2, d_3, etc., of

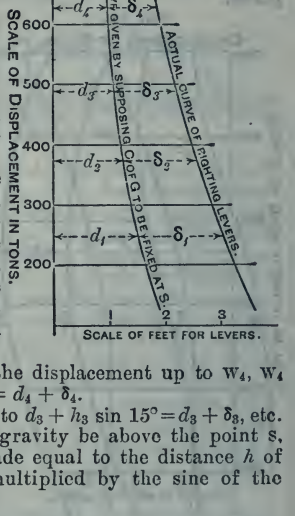
the centre of buoyancy up to each inclined water-plane from the vertical AB.

By assuming the centre of gravity to be at S, and fixed there for all draughts, the distances d_1, d_2, d_3, etc., would be the righting levers at the displacements, up to the respective water-planes W_1, W_2, W_3, W_4.

These lengths are then set off as ordinates along an axis having the several displacements up to the water-planes as abscissæ.

The actual righting levers can then be determined, when the correct positions of the centres of gravity corresponding to the various displacements are fixed, by multiplying the respective distances h_1, h_2, h_3, etc., of the actual centres of gravity g_1, g_2, g_3, g_4 below S by the sine of the angle of heel, and adding this length to the arms already found (see fig. 128).

FIG. 128.



The actual righting lever for the displacement up to W_4, W_4 would be equal to $d_4 + h_4 \sin 15° = d_4 + \delta_4$.

Up to W_3, W_3 it would be equal to $d_3 + h_3 \sin 15° = d_3 + \delta_3$, etc.

Should any of the centres of gravity be above the point S, a deduction would have to be made equal to the distance h of the centre of gravity above S multiplied by the sine of the angle of heel.

To Calculate the Statical and Dynamical Stabilities of a Vessel at Successive Angles of Heel.

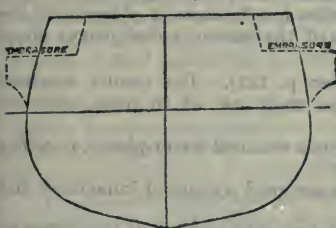
Among the various methods that are used for calculating the statical and dynamical levers, three are here described—(*a*) Barnes' method, (*b*) the direct method, (*c*) the integrator method; the last named is by far the quickest and most convenient. Either equidistant sections may be employed using Simpson's rule or specially spaced sections with Tchebycheff's rule (see displacement sheet, p. 100). The former may obviate the preparation of a special body plan, but the latter rule is generally more expeditious on the whole. It is generally assumed that all weights are fixed, all openings in the sides and decks closed and made watertight, all appendages can be neglected, and that no change of trim takes place during inclination.

Barnes' Method.

1. *Body Plan.*—Prepare a body plan (fig. 130) in which all the sections are taken perpendicular to the load water-line,

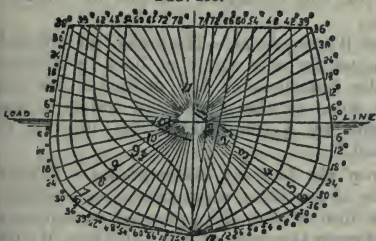
and at equal distances apart (if Tchebycheff's method be employed the sections are spaced as shown in the displacement sheet, p. 100). In constructing it the sections should be made fair continuous curves, any irregularities which might be caused by embrasures, etc., being left out (as shown in full lines in fig. 129, where the dotted lines show the actual section of vessel), they being treated separately afterwards as *appendages*. When there are appendages it is also necessary to have correct sheer and half-breadth draughts, in order to calculate their volume, etc.

FIG. 129.



2. *Angular Interval.*—The body plan has now to be crossed by a number of lines, radiating from the middle point of the load water-plane, and at equiangular intervals from 6° to 10°, arranging if possible that one passes through the edge of the upper continuous deck amidships.

FIG. 130.



The equiangular interval is determined as follows :—Divide the angle which the radiating line, passing through the edge of the upper deck, makes with the load water-line into such a number of equiangular intervals that the line passing through the edge of the upper deck becomes a stop-point in the integration to which these radiating lines will be afterwards treated. If Simpson's first rule is used the number of intervals must be even ; if his second rule, a multiple of three must be used, and so on.

3. *Measuring the Ordinates.*—The ordinates of the immersed and emerged sides of the various inclined longitudinal water-planes are measured off right fore and aft for each successive angle of heel from the middle line of the ship, and entered upon a set of tables, styled *preliminary tables*, under their proper heading. One of these tables is necessary for each separate angle of heel.

4. *Preliminary Table.*—Three operations are performed (see p. 122) upon the ordinates entered in these tables. Firstly, they are affected by a set of Simpson's multipliers, in order

to find a function for the *area* of the immersed and emerged sides of the respective radial planes. Secondly, the squares of the ordinates are affected by the same set of multipliers in order to find a function for the *moment* of the immersed and emerged sides of the respective radial planes. Thirdly, the cubes of the ordinates are affected by the same set of multipliers in order to find a function for the *moment of inertia* of the immersed and emerged sides of the various radial planes about the middle line of ship.

5. *Combination Tables* (see p. 123).—The results obtained in the preliminary tables are made use of in these tables to determine—

(1st) The area of the various inclined water-planes, together with their centres of gravity.

(2nd) The volumes of the assumed wedges of immersion and emersion.

(3rd) The position of the true water-planes at the different angles of heel.

(4th) The moments of the corrected wedges of immersion and emersion.

6. *Areas of the Inclined Water-planes.*—The area of an inclined water-plane is easily found for any angle of heel by adding together the sums of the functions of the ordinates for the immersed and emerged sides of the respective water-planes, and multiplying the result by $\frac{1}{3}$ the longitudinal interval if Simpson's first rule is used.*

7. *Centre of Gravity of the Inclined Water-planes.*—To find the distance of the centre of gravity of any inclined water-plane relatively to the middle line of the ship, proceed as follows:—Take the difference between the sums of the functions of the squares of the ordinates for the immersed and emerged sides of the water-plane; divide the result by 2 and multiply the quotient by $\frac{1}{3}$ the longitudinal distance between the ordinates, if Simpson's first rule is used. That product divided by the area of the water-plane will give the distance of its centre of gravity from the middle line.

8. *Volumes of Assumed Wedges.*—Take the sums of the functions of the squares of the ordinates for both sides of each of the radial planes contained in the wedges of immersion and emersion, and enter them in their proper column in the combination table, and affect them by a proper set of multipliers; add their results together, subtract the lesser sum from the greater, and divide the result by 2. The quotient multiplied by $\frac{1}{3}$ the longitudinal distance between the ordinates, if Simpson's first rule is used (this division by 3 is generally done in the preliminary tables): this final product multiplied by $\frac{1}{3}$ of the equiangular interval in circular measure, if Simpson's first rule is again

* *Note.*—The division by 3 is generally done in the preliminary tables.

used, will give the difference between the volumes of the assumed wedges of immersion and emersion. If there are any appendages the necessary additions or deductions are made here.

9. *Correcting Layer.*—If the volume of the assumed wedge of immersion exceeds that of the wedge of emersion, it shows that the displacement up to the radial plane is too great, and that to find the true water-plane a parallel layer must be taken away from the assumed wedges; but if the wedge of emersion exceeds that of immersion, a parallel layer must be added to the wedges.

The *thickness* of this layer is found by dividing the difference between the volumes of the two assumed wedges by the area of the proper radial water-plane, having made any additions or deductions in the case of appendages.

10. *Moments of Wedges for Statical Stability.*—The sums of the functions of the cubes of the ordinates for both the immersed and emerged wedges are placed in the proper column in the combination table, and are affected by the same set of multipliers as were determined for the sums of the functions of the squares; the products are multiplied by the various cosines of the angles of inclination made by the radial planes with the load water-line; the products are then added together and the sum divided by 3; the quotient is then multiplied by $\frac{1}{3}$ the angular interval, and that product by $\frac{1}{3}$ the longitudinal interval, between the ordinates, if Simpson's first rule has been used (this division by 3 is generally done in the preliminary tables): the final result will be the moment of the wedges about a line perpendicular to the radial plane, and passing through the middle point of the load water-plane. The corrections for the moments of the appendages must now be added or subtracted, as the case may be, also the correction for the layer, if any, must be done here, its moment being found by multiplying its volume by the distance of the centre of gravity of its water plane from the middle point of the load water-plane. If the centre of gravity of the layer lies towards that side for which the assumed wedge is the greater, the correction must be deducted; if it lies towards the opposite side, it must be added. This final result, being divided by the total volume of displacement, will give the length of the arm BR (see fig. 119). Multiply the height of the centre of gravity above the centre of buoyancy by the sine of the angle of heel, and subtract the product from BR; the remainder will be the length of the arm of the righting couple GZ; GZ multiplied by the displacement in tons will give the righting moment, or statical stability, of the ship for the given angle of heel.

11. *Moments of the Wedges for Dynamical Stability.*—This result is determined in a manner somewhat similar to that pursued for the statical stability, the only difference being that the

PRELIMINARY TABLE FOR STABILITY AT 30° ANGLE OF HEEL.

Nos. of Secs.	Ordinates	Multipliers	Functions of Ordinates	Squares of Ordinates	Multipliers	Functions of Squares	Cubes of Ordinates	Multipliers	Functions of Cubes
			IMMERSED WEDGE.						
1	·8	½	·4	·6	½	·3	·5	½	·3
1½	8·1	2	16·2	65·6	2	131·2	531·4	2	1062·8
2	14·2	1	14·2	201·6	1	201·6	2863·3	1	2863·3
2½	17·8	2	35·6	316·8	2	633·6	5639·7	2	11279·4
3	20·5	1½	30·7	420·2	1½	630·3	8615·1	1½	12922·7
4	20·4	4	81·6	416·2	4	1664·8	8489·7	4	33958·8
5	20·2	2	40·4	408·0	2	816·0	8242·2	2	16484·4
6	20·2	4	80·8	408·0	4	1632·0	8242·2	4	32969·6
7	20·2	2	40·4	408·0	2	816·0	8242·2	2	16484·4
8	20·2	4	80·8	408·0	4	1632·0	8242·2	4	32969·6
9	20·2	1½	30·3	408·0	1½	612·0	8242·2	1½	12363·6
9½	20·3	2	40·6	412·0	2	824·0	8363·6	2	16727·2
10	18·8	1	18·6	353·4	1	353·4	6644·7	1	6644·7
10½	15·8	2	31·6	249·6	2	499·2	3944·3	2	7888·6
11	10·6	½	5·3	112·4	½	56·2	1191·0	½	595·5
			3)547·3			3)10502·6			3)204972·9
			182·4			3500·9	Immersed		68324·3
							Emerged		58590·4
							Both wedges		126914·7
			EMERGED WEDGE.						
1	1·1	½	·5	1·2	½	·6	1·3	½	·7
1½	6·5	2	13·0	42·2	2	84·4	274·6	2	549·2
2	10·9	1	10·9	118·8	1	118·8	1295·0	1	1295·0
2½	14·1	2	28·2	198·8	2	397·6	2803·2	2	5606·4
3	16·9	1½	25·3	285·6	1½	428·4	4826·8	1½	7240·2
4	20·0	4	80·0	400·0	4	1600·0	8000·8	4	32003·2
5	21·2	2	42·4	449·4	2	898·8	9528·1	2	19056·2
6	21·5	4	86·0	462·2	4	1848·8	9938·4	4	39753·6
7	21·2	2	42·4	449·4	2	898·8	9528·1	2	19056·2
8	20·1	4	80·4	404·0	4	1616·0	8120·6	4	32482·4
9	17·5	1½	26·2	306·2	1½	459·3	5359·4	1½	8039·1
9½	15·4	2	30·8	237·1	2	474·2	3652·3	2	7304·6
10	12·5	1	12·5	156·2	1	156·2	1953·1	1	1953·1
10½	8·9	2	17·8	79·2	2	158·4	705·0	2	1410·0
11	3·5	½	1·7	12·2	½	6·1	42·8	½	21·4
			3)508·1			3)9146·4			3)175771·3
			169·3			3048·8			58590·4

COMBINATION TABLE FOR STABILITY AT 30° ANGLE OF HEEL.

IMMERSED WEDGE

Angles of Heel	Functions of Ordinates	Functions of Squares of Ordinates	Multis.	Products
0°	—	3015·4	1	3015·4
5°	—	3098·2	4	12392·8
10°	—	3173·2	2	6346·4
15°	—	3325·4	4	13301·6
20°	—	3515·9	2	7031·8
25°	—	3773·3	4	15093·2
30°	182·4	3500·9	1	3500·9
Immersed Wedge . . .				60682·1
Emerged Wedge . . .				54148·8
				2) 6533·3
				3266·7
⅓ Angular Interval . .				·0291
				32667
				294003
				65334
				95·06097
Longitudinal Interval . .				20
Excess of Immersed Wedge				1901·22
Appendage				396·00
Area of Water-plane .			7197	2297·22
Thickness of Layer . .				·32

CORRECTING LAYER

Moment for Statical Layer
=2297×1·16=2664.
Moment for Dynamical Layer
=2297×·16=367

EMERGED WEDGE

Functions of Ordinates	Functions of Squares of Ordinates	Multipliers	Products
—	3015·4	1	3015·4
—	2976·3	4	11905·2
—	2974·1	2	5948·2
—	2991·3	4	11965·2
—	3031·0	2	6062·0
—	3051·0	4	12204·0
169·3	3048·8	1	3048·8
			54148·8

INCLINED WATER-PLANE

Functions of ordinates	Immersed side	182·4
	Emerged side .	169·3
		352·7
Longitudinal Interval . .		20
Uncorrected Area . . .		7034
Appendage		163
Total Area		7197
Functions of squares	Immersed side .	3500·9
	Emerged side .	3048·8
		2) 452·1
		226·05
Longitudinal Interval . .		20
Uncorrected Moment . .		4521
Moment for Appendage .		2818
Area of Water-plane .		7197
		8339
C. of G. towards immersed side		1·16

BOTH WEDGES

Statical Stability

Sums of Functions of Cubes	Multipliers	Products	Cosines of Heel	Products for Moments
111478·2	1	111478·2	·8660	96540·1
112940·9	4	451763·6	·9063	409433·4
115141·4	2	230282·8	·9397	216396·7
130230·2	4	480920·8	·9659	464521·4
127258·5	2	254517·0	·9848	250648·3
135895·3	4	543581·2	·9962	541515·6
126914·7	1	126914·7	1·0000	126914·7
				3)2105970·0
				7019900
⅓ Angular Interval				·0291
				7019900
				63179100
				14039800
				204279900
Longitudinal Interval . .				20
Moment of Wedges uncorrected				4085598·18
Correction for appendages .				8939·00
,, ,, layer . .				−2664·00
Volume of Displacement .			86767	4115044·18
BR=				4·78
BG . sine 30°= . . .				3·43
GZ=Statical Lever= . .				1·65

Dynamical Stability

Sines of Heel	Products for Moments
·5000	55739·1
·4226	190503·8
·3420	78756·7
·2588	124462·3
·1736	44184·2
·0871	47345·9
·0000	·0
	3)541392·0
	180464·0
⅓ Angular Int.	·0291
	1804640
	16241760
	3609280
	5251·50240
Long. Int.	20
	105030·0 18
	1547·000
	−367·000
86767)	106210·048
B_1R= . .	1·224
BG . vers 30°=	·839
Dynam. Lever=	·385

APPENDAGES

Pieces	Area	Mo. of Area	Vols.	Mo. of Vols.
Battery	150	3495	650	14361
Stern Im.	38	130	58	278
Stern Em.	12	−12	−12	−36
Negative	−37·5	−683·7	−300	−5700
Totals	162·5	2818·3	396	8939

Moment of Statical Stability in foot tons
=2479×1·65=4090
Moment of Dynamical Stability in foot tons
=2479×·385=954

Longitudinal Int.=20 feet. Angular Int.=5 deg. C. of Grav. above C. of Buoy.=6·26. Vol. of Displacement=86767. Displacement in tons=2479.

sums of the functions of the cubes are multiplied by the sines of the various angles of inclination instead of the cosines; the sum of the products so obtained being divided and multiplied by the same numbers as were used for the statical stability, in order to find the moment of the wedges uncorrected relatively to the respective radial planes. The corrections for the appendages are then made, that for the correcting layer being subtracted in all cases. The moment for the correcting layer is found by multiplying its volume by half its thickness, that being about the vertical height of its centre of gravity from its radial plane. This final result divided by the total volume of displacement will give the length of the arm B′R, from which if BG . vers θ be deducted, the remainder will equal the length of the arm for the dynamical stability, or the vertical height through which the centre of gravity of the ship has been lifted and the centre of buoyancy depressed.

12. *Geometrical Mode of Calculating Dynamical Stability.*—The dynamical stability of a vessel at any given angle of heel is the sum of the moments of the statical stability taken at indefinitely small equiangular intervals up to the given angle of heel, and is therefore equal to the area of the curve of statical stability included between the origin of the curve and the angle in question. It must be noticed that the abscissæ of a curve of statical stability is given in angles, and therefore the longitudinal interval is taken in circular measure.

But, as the lengths of the arms for statical stability are generally used to construct a curve instead of the moments of stability, the area, as above found by the rule from such a curve, will necessarily give the length of the *arm* for dynamical stability and not the *moment*.

Example (see fig. 123).—To find the length of the arm for dynamical stability at an angle of 30° inclination.

Angles of Heel	Lengths of Statical Levers GZ	Simpson's Multipliers	Products
0 degrees	·0	1	·0
5 „	·2	4	·8
10 „	·42	2	·84
15 „	·68	4	2·72
20 „	·97	2	1·94
25 „	1·30	4	5·20
30 „	1·66	1	1·66
			13·16

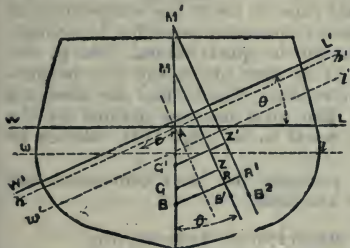
$\frac{1}{3}$ of angular interval in circular measure = ·0291

1316
11844
2632

Dynamical lever for 30° = ·382956

13. *Curve of Stability for Light Draught.*—The lengths of the arms for this curve can readily be approximated from the results obtained for the curve in the load condition.

FIG. 131.



In fig. 131 WL is the load water-line, and *wl* the light water-line, for the upright position of the vessel. If the vessel is inclined through an angle θ, and W'L' is the true position of the inclined water-plane for the load condition, then the true position of the water-plane for the light condition will run parallel to it, as *w'l'*. To determine its perpendicular distance from W'L', divide the volume of the layer contained between the light and load water-planes by the area of the assumed inclined water-plane *hh'*, which was found for the inclined load condition. Let B be the centre of buoyancy for the upright load condition, B' for the inclined load condition, and B^2 for the inclined light condition. BR is perpendicular to the vertical B'M, and BR' is perpendicular to the vertical B^2M'.

Let v equal volume of light displacement.

„ v = volume of displacement contained between the light and load water-planes.

„ c = distance of centre of gravity of assumed inclined water-plane from the vertical through A, assumed positive on the emerged side.

„ GZ and G'Z' = the lengths of the arms of the righting couples for the load and light condition respectively.

$$\text{Then } G'Z' = GZ - GG' \sin\theta + \frac{v(BR - BA\sin\theta + c)}{V}$$

Example.—In the ship illustrated in the tables (pp. 122, 123) find the lever of statical stability at 30° when light, assuming the displacement diminished by 200 tons, and the c.g. raised by 1·5 ft. B is 6·5 ft. below original upright W.L.

Here $v = 200 \times 35 = 7000$; $V = 86767 - 7000 = 79800$ approx.

GZ = 1·65; GG' = 1·5; $\sin\theta = \frac{1}{2}$; BR = 4·78; BA = 6·5; $c = -1{\cdot}16$.

$$GZ' = 1{\cdot}65 - {\cdot}75 + \frac{7000(4{\cdot}78 - 3{\cdot}25 - 1{\cdot}16)}{79800} = {\cdot}93'.$$

DIRECT METHOD.

Lay a piece of tracing paper over the body plan, and on it draw a trial water-line at the correct inclination. Trace the

wedge sections, replacing the curved portions by one or two straight lines approximating as closely as possible to the curves. Find graphically the areas of the triangles and quadrilaterals, and thence determine the volume of each wedge. If these are not nearly equal raise or lower the water-line, and proceed as before until there is practical equality in volume. Find the e.g. of each triangle or quadrilateral (see p. 59) and calculate the moment of its area about any line perpendicular to the inclined water-line. Thence find the moments of the volumes and add. The total moment divided by the volume of displacement is equal to BR (fig. 119), whence GZ is at once determined.

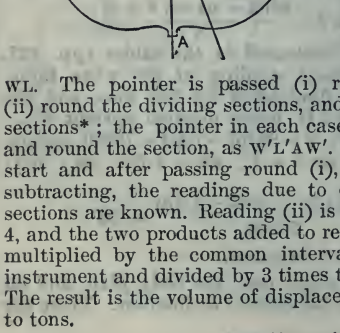
The direct method is, perhaps, the most convenient one when an integrator is not available.

Amsler-Laffon's Mechanical Integrator.

By means of this instrument, the area, moment, and moment of inertia about any axis, can be obtained for any curvilinear area by tracing its outline with a pointer.

Its principal use is that of obtaining the stability of a vessel at various angles of heel and at various drafts. It is usual, when using this instrument, to first calculate the righting lever for a number of displacements at *one* inclination, say 15°. Then the same for 30°, 45°, and so on; the cross curves being constructed before the ordinary curves.

Fig. 132



Let fig. 132 be a body plan drawn for both sides of a ship; let WL be its upright waterline intersecting the middle line at S. Through S draw inclined waterlines at the required inclinations, and let W'L' be any one of them, say at 15°. The first step is to find the displacement at W' L' as it is generally different from that at WL. The pointer is passed (i) round the two end sections, (ii) round the dividing sections, and (iii) round the intermediate sections* ; the pointer in each case passing along the waterline and round the section, as W'L'AW'. Readings are taken at the start and after passing round (i), (ii), and (iii), so that after subtracting, the readings due to each of the three series of sections are known. Reading (ii) is multiplied by 2, and (iii) by 4, and the two products added to reading (i). The total is then multiplied by the common interval and the constant of the instrument and divided by 3 times the square of the scale used. The result is the volume of displacement, which is then reduced to tons.

* See Simpson's Rules.

If in the same way ST, the line through S, perpendicular to W′L′ is made the axis for moments, and the readings for moments are treated in the same way as those for areas, it is evident that the final result will be the moment of the underwater portion about ST as axis (obviously, the total must now be divided by 3 times the *cube* of the scale instead of the square). This divided by the volume of displacement will give the perpendicular distance of the inclined centre of buoyancy from ST; that is SZ, when B′Z is parallel to ST.

The righting lever, or GZ, is equal to $SZ + SG \sin\theta$ when G is below S as at G_1; and equal to $SZ - SG \sin\theta$ when G is above S.

The righting lever GZ is set off at its proper displacement on the cross curve for 15°. This is done at different waterlines and the cross curve thus completed.

The following is the actual form of the calculation for SZ.

Sections 10·6 apart. Scale of body $\frac{1}{4}''$ to 1 foot.

Machine constants { Areas $\frac{20}{1000}$. Moments $\frac{40}{1000}$.

ANGLE OF HEEL 15°

Sections	Areas				Moments			
	Readings	Differences	Simpson's Multipliers	Products	Readings	Differences	Simpson's Multipliers	Products
Initial . .	4029	—	—	—	982	—	—	—
End ordinates .	4111	82	1	82	986	4	1	4
Dividing ordinat.	10502	6391	2	12782	1398	412	2	824
Intermediate „ .	17309	6807	4	27228	1819	421	4	1684
				40092				2512

Displacement in tons

$$= 40092 \times \frac{20}{1000} \times (4)^2 \times \frac{10{\cdot}5}{3} \times \frac{1}{35} = 1283.$$

$$SZ = \frac{2512}{40092} \times \frac{40}{20} \times 4 = {\cdot}5 \text{ feet.}^*$$

* The 4 multiplier is the reciprocal of the scale of the drawing.

Tchebycheff's rule (see p. 43) can be very usefully employed instead of Simpson's rule in the above; the saving of time due to its adoption is, for a complete set of cross curves, more than sufficient to compensate for the time of preparing the special body plan, which need only be drawn in fairly roughly. In this labour may be avoided by using the sections numbered 2, 5, 7, 10, 12, 15, 17, and 20 from an ordinary body plan whose equidistant sections are numbered 1 to 21. It will be found that these coincide nearly in position with those required with Tchebycheff's rule for 4 ordinates, repeated. This was pointed out at Inst. N.A., 1915, by Mr. W. J. Luke.

Example.—Length of ship, 210 feet; number of sections, 8; scale of body, $\frac{1}{4}$" to 1 foot. Machine constants as before.

$$\text{Displacement in tons} = \frac{20}{1000} \times 16 \times \frac{210}{8} \times \frac{1}{35} \times \text{area reading}.$$

$$\text{SZ} = \frac{\text{Moment reading}}{\text{Area reading}} \times 8 \qquad \begin{matrix} 4 = \text{the scale} \\ 2 = \text{ratio of} \\ \text{machine con-} \\ \text{stants.} \end{matrix}$$

Note.—The above or 'all-round' method is the simplest, since it gives directly the stability lever desired. A more accurate and expeditious method, however, is that known as the 'figure-eight'. The pointer is passed around the outline of the wedge sections, only, taking them in the opposite directions on the two sides of the ship; e.g. commencing at S (fig. 132) the pointer reaches the points L′, L, S, W, W′, S in the order named. The result is to give the difference of the wedge volumes (by the area reading) and the sum of their moments (by the moment reading). If V be the original volume of displacement, v the increased volume registered by the machine, and M the moment registered (the last two being found from the readings as in the 'all-round' method), and BS the distance of the upright C.B. below S,

$$\text{SZ} = \frac{\text{M} - \text{V} \cdot \text{BS} \sin\theta}{\text{V} + v}$$

FORMULÆ FOR STABILITY LEVERS IN SPECIAL CASES.

1. *Ship with concentric circular sections, cylinder.*—The metacentric method is here applicable to all angles of heel and statical lever GZ = GM sin θ, dynamical lever = GM vers θ.

2. *Wholly immersed vessel.*—The metacentre and centre of buoyancy are coincident, and the above formulæ apply if B be substituted for M.

3. *Wall-sided vessel, parabolic cylinder.*—Statical lever GZ = sin θ (GM + $\frac{1}{2}$ BM tan$^2\theta$).

Dynamical lever = GM (1 − cos θ) + $\frac{1}{2}$ BM (sec θ + cos θ − 2).

For BM, its value when ship is upright is intended.

Change of Metacentric Height due to Small Changes in Dimensions.

Let the beam of a ship be increased by $\frac{1}{n_1}$ of itself, all transverse ordinates being augmented in the same proportion. Similarly let the draught be increased by $\frac{1}{n_2}$ of itself. If these changes are moderate, and the height of the c.g. above the keel be assumed to vary as the draught, the increase of metacentric height is given by—

$$\frac{\delta m}{m} = \frac{2}{n_1}\left(1 + \frac{a}{m}\right) - \frac{1}{n_2}\left(1 + \frac{2a}{m}\right) \quad . \quad . \quad . \quad (1)$$

where m is the original GM, δm the increase of GM, and a is BG.

If the beam only be increased, $\frac{1}{n_2} = 0$, and

$$\frac{\delta m}{m} = \frac{2}{n_1}\left(1 + \frac{a}{m}\right) \quad . \quad . \quad . \quad (2)$$

If the draught be diminished so as to maintain the same displacement as before, $\frac{1}{n_2} = -\frac{1}{n_1}$, and

$$\frac{\delta m}{m} = \frac{1}{n}\left(3 + \frac{4a}{m}\right) \quad . \quad . \quad . \quad (3)$$

In the preceding case if the total depth be unaltered (the freeboard being increased to compensate for the diminution of draught), and if μ represent the height of the c.g. above the keel, originally,

$$\frac{\delta m}{m} = \frac{1}{n}\left(3 + \frac{4a - \mu}{m}\right) \quad . \quad . \quad . \quad (4)$$

If in the preceding case it be assumed alternatively that the freeboard is unaltered (the height of c.g. above keel varying as the total depth as before), and if s represent the original ratio — freeboard ÷ draught,

$$\frac{\delta m}{m} = \frac{1}{n}\left\{3 + \frac{1}{m}\left(4a - \frac{\mu s}{1+s}\right)\right\} \quad . \quad . \quad . \quad (5)$$

In the general case, determine the effect on GM of increasing the beam by one foot, assuming that $BM \propto LB^3/\Delta$. The increase of GM roughly varies as that of beam.

Example 1.—In making a preliminary estimate of the dimensions of a new design, the beam is assumed 36 feet, the distance BG is 8 feet and GM is 1 foot. It is desired to double the metacentric height, while maintaining the draught unaltered. Find the beam required.

Using formula (2), $\delta m = 1$, $m = 1$, $a = 8$.

Whence $\frac{2}{n_1}(1 + 8) = 1$, or $\frac{1}{n} = \frac{1}{18}$.

Therefore the beam required is 36 $(1 + \frac{1}{18})$ or 38 feet.

Note that if it is desired not to alter the displacement, the length must be diminished by $\frac{1}{18}$ of itself.

Example 2.—In a battleship having beam 89 feet, mean draught 27 feet, GM 5 feet, G above water-line $6\frac{1}{2}$ feet, and BG 18 feet, find the effect on the metacentric height of increasing the beam by 1 foot, assuming that owing to a change in the distribution of weights the c.g. is 0·35 feet higher above the water-line in the new design. The displacement and length are assumed unaltered.

Using formula (3), $m = 5$, $n_1 = \frac{1}{89}$, $a = 18$.

Whence $\frac{\delta m}{5} = \frac{1}{89}\left(3 + \frac{72}{5}\right) = \cdot 195$ or $\delta m = \cdot 98$ feet.

But this assumes that the height of G above water-line becomes $6 \cdot 5\left(1 + \frac{1}{n_2}\right)$ or $6 \cdot 5\left(1 - \frac{1}{89}\right)$ or 6·43 feet; it is actually 6·85 feet.

Hence the metacentric height is $5 + \cdot 98 - (6 \cdot 85 - 6 \cdot 43) = 5 \cdot 56$ feet.

Alteration of Stability Curve due to Small Changes in Dimensions.

Assume the beam increased by $\frac{1}{n_1}$ of itself, and the draught by $\frac{1}{n_2}$ of itself as above. This process is applicable to any two ships of fairly similar form, even if they depart somewhat from exact proportionality. Given the curve of statical stability (GZ) for the first ship, it is required to construct the corresponding curve for the desired vessel without constructing the body plan or performing the usual calculations.

Rule.—1. Construct the curve of dynamical stability of the first ship by taking areas of the GZ curve (see p. 116). Two or three spots are sufficient, as great accuracy is not required.

2. Corresponding to the angle θ at which the stability lever is required in the new ship, determine an angle ϕ from the formula—

$$\left(1 + \frac{1}{n_2}\right) \tan \phi = \left(1 + \frac{1}{n_1}\right) \tan \theta.$$

3. Determine GZ, the statical lever, and z the dynamical stability lever for the original ship at the angle ϕ.

4. Determine δm, the increase of metacentric height, by the methods of the previous page.

5. Then the stability lever G′Z′ of the new ship at the angle θ is given by—

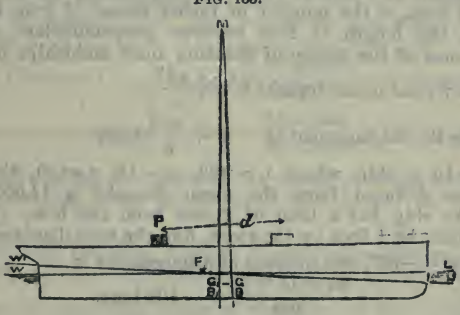
$$\text{G}'\text{Z}' - \text{GZ} = \delta m \sin \theta + \tfrac{1}{2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)(\text{GZ} - m \sin \phi)$$
$$+ \tfrac{1}{2}\left(\frac{1}{n_1} - \frac{1}{n_2}\right)(\text{GZ} \cos 2\phi + 2 \sin 2\phi\,(a + z) - (3m + 4a) \sin \phi).$$

By calculating G′Z′ for about 3 values of θ, the stability curve can be described by the aid of the tangent at the origin as given by the GM.

Longitudinal Stability.

Definitions.—1. The *centre of flotation* is the centre of gravity of the water-plane; it is denoted by F in fig. 133. For longitudinal inclinations without change of displacement the water-planes intersect on a transverse axis passing through the centre of flotation.

Fig. 133.



2. The difference between the draught forward and that aft is termed the *trim*. If the former is greater the trim is by the bow, and vice-versa. When not stated the draughts are supposed taken at the perpendiculars; they are actually measured at the draught marks which are frequently placed at the extremities of the straight keel.

3. *Change of trim* is the sum of the changes of draught forward and aft if one is increased and the other diminished; otherwise it is the difference between the changes of draught.

To determine the draughts and trim at the draught marks given those at the perpendicular, and the converse.

Let L = length of ship between perpendiculars.

a, b = distance of forward and after draught marks from amidships.

D_1, D_2 = draughts at F.P. and A.P.

D_3, D_4 = draughts at forward and after draught marks.

$$D_3 = \frac{D_1 + D_2}{2} + \frac{a}{L}(D_1 - D_2).$$

$$D_4 = \frac{D_1 + D_2}{2} + \frac{b}{L}(D_2 - D_1).$$

$$D_4 - D_3 = \frac{a+b}{L}(D_2 - D_1).$$

$$(a+b)D_1 = D_3\left(\frac{L}{2}+b\right) - D_4\left(\frac{L}{2}-a\right)$$

$$(a+b)D_2 = D_4\left(\frac{L}{2}+a\right) - D_3\left(\frac{L}{2}-b\right)$$

To determine the displacement of a vessel floating out of her designed trim.

Let D be mean draught amidships, W the corresponding displacement as obtained from the displacement sheet, T the tons per inch, d the number of inches excess of trim by the stern, L the length in feet between perpendiculars, and c the distance of the centre of flotation abaft amidships in feet.

Then virtual mean draught is $D + \frac{cd''}{L}$

Hence the displacement is $W + \frac{T\,cd}{L}$ tons.

Ex.—In a ship where $L = 400$, $c = 15$, $T = 80$, the displacement deduced from the mean draught is 14,000 tons where the ship has a trim of 2 feet from the bow. If the normal trim be 1 foot by the stern, find the true displacement.

Here $d = -36''$, and increase of displacement is

$$-\frac{80 \times 15 \times 36}{400} = -108 \text{ tons.}$$

Hence displacement is $14{,}000 - 108 = 13{,}892$ tons.

Note.—1. The distance c expressed as a fraction of the ship's length has the following average values :—Battleship $\frac{1}{18}$, light cruiser $\frac{1}{12}$, T.B. destroyer $\frac{1}{15}$, steam yacht $\frac{1}{30}$, channel steamer, $\frac{1}{30}$, cargo steamer $\frac{1}{100}$.

2. The centres of buoyancy and gravity lie abaft the midship section at a distance, which, expressed as a fraction of the ship's length, has the following average values :—Battleship $\frac{1}{25}$, light cruiser $\frac{1}{25}$, T.B. destroyer $\frac{1}{30}$, steam yacht $\frac{1}{70}$, channel steamer $\frac{1}{50}$, cargo steamer 0.

3. For a change of trim t without change of displacement, the draught forward is altered by $\frac{t}{2} + \frac{tc}{L}$ and that aft by $\frac{t}{2} - \frac{tc}{L}$

To find the changes of draught and trim in passing from salt to fresh water, and vice versa.

Let the symbols W, T, and D above refer to salt water. Let δ inches be the sinkage in fresh water, and D′ the final mean draught.

Then $D' = D + \delta/12$; $\delta = \frac{\cdot 9W}{35T} = \frac{W}{38 \cdot 9T}$

It is assumed above that the fresh water occupies 35·9 cubic

feet to the ton. If the water is brackish, and occupies $35 + x$ cubic feet to the ton, the latter formula becomes

$$\delta = \frac{x\text{W}}{35\text{T}}$$

The change of trim is usually very small. If c' be the distance of the centre of flotation abaft the centre of buoyancy, and M the moment to change trim in salt water, the change of trim by the bow on passing from salt to brackish water is, in inches, $\frac{x c' \text{W}}{35\,\text{M}}$; or $\frac{c'\text{W}}{38{\cdot}9\text{M}}$ for fresh water where x is ·9.

Ex.—Find the changes of draught and trim in a light cruiser on passing into fresh water if $\text{W} = 3000$, $\text{T} = 25$, $\text{M} = 650$, $c' = 11$.

$$\text{Increase of mean draught} = \frac{3000}{38{\cdot}9 \times 25} = 3{\cdot}1 \text{ inches.}$$

$$\text{Change of trim by the bow} = \frac{11 \times 3000}{38{\cdot}9 \times 650} = 1{\cdot}3 \text{ inches.}$$

To determine the positions at which a weight must be added or removed so as to leave the draught at one end constant.

RULE.—To maintain constant draught at a distance y abaft (*or before*) the centre of flotation, place the weight at a distance x before (*or abaft*) the centre of flotation, where $x = \frac{\text{ML}}{\text{T}y}$. If constant draught is desired at either perpendicular, the two points for the weight are situated at a distance $\frac{2\,\text{M}}{\text{T}}$ very nearly from the C.F. This distance is about $\frac{60\,\text{T}}{\text{B}}$, or about $\frac{\text{L}}{10}$ in many ships.

To determine the vertical height of the longitudinal metacentre above the centre of buoyancy.

Divide the moment of inertia of the water-plane relatively to a transverse axis passing through the centre of flotation by the volume of displacement (for example, see displacement sheet and explanation on pp. 94, 99).

Note.—This height is frequently greater than the ship's length, so that BG is negligible in comparison; then $\text{GM} = \text{BM}$ approximately.

Moment to alter trim of a vessel.—In fig. 133 let the weight P be moved forward through a longitudinal distance l, changing the water-line from WL to W′L′.

Then $\text{P}d = \text{W}.\text{GG}_1 = \frac{\text{W}.\text{GM}}{\theta} = \frac{\text{W} \times \text{GM} \times \text{trim in feet}}{\text{L}}$; hence trim in inches $= \frac{12\,\text{P}d\,\text{L}}{\text{W} \times \text{GM}}$

The product Pd is the moment causing trim; if several weights are moved, their moments are added, allowing for sign. Note that the *moment to change trim one inch* is equal to the expression $\frac{W \times GM}{12L}$. This is fairly constant within moderate changes of draught, and practically unaffected by vertical shifts of weight.

Approximate formulæ.

1. (J. A. Normand). Long. BM $= 13{,}000 \frac{T^2L}{BV}$, L = length on L.W.L in feet, B = beam in feet, V = volume of displacement in cubic feet, T = tons per inch.

2. (Derived from the preceding)
$$\text{Moment to change trim } 1'' = 30 \frac{T^2}{B}$$

3. Moment to change trim $1'' = L^2B/10{,}000$ in ships of ordinary form, $TL \div 18{\cdot}5$.

EFFECT OF ADDING WEIGHTS OF MODERATE AMOUNT.

The weights added are supposed insufficient to affect appreciably the transverse stability, or to cause relatively large heel, trim, or immersion.

RULE.—Find the distance of each weight from the middle line plane and from amidships. Calculate the moments, positive and negative (weights removed are considered negative), and add

$$\text{Mean sinkage} = \text{weight added} \div \text{tons per inch.}$$

$$\text{Heel in degrees} = 57{\cdot}3 \times \frac{\text{transverse moment}}{\text{displacement} \times GM}$$

$$\text{Trim in inches} = \frac{\text{longitudinal moment about centre of flotation}}{\text{moment to change trim 1 inch.}}$$

$$= \frac{\text{longitudinal moment about } \mathbb{X} \pm (w \times \text{C.F. abaft } \mathbb{X})}{\text{moment to change trim 1 inch}}$$

using + sign when the net weight (w) added or subtracted is before amidships.

EFFECT OF ADDING WEIGHTS OF CONSIDERABLE AMOUNT.

RULE.—Add the weights and their moments as above, including in addition the vertical moments required to find the rise or fall of the c.g.

The new mean draught is found from the curve of displacement, or more accurately from the curve of tons per inch, by estimating the mean tons per inch between the two waterlines. If necessary make the correction due to the position of the centre of flotation as described on p. 132.

To obtain the heel find first the vertical position of G; from the metacentric diagram the new GM is obtained. The latera

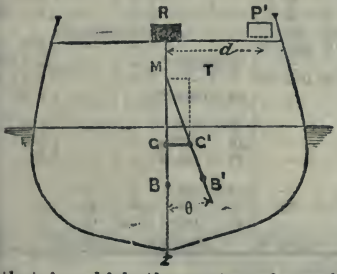
movement (GG′) of G is found by dividing the transverse moment by the new displacement. A moderate angle (θ) of heel is given by the formula $\tan \theta = GG'/GM$. If θ is very large, construct a curve of stability for the new condition, using the cross curves, and find by trial the angle θ at which the relation $GG' = GZ \sec \theta$ holds.

For the trim the method given on the preceding page is usually sufficiently accurate. If, however, the sinkage is very great, construct a curve of moment to change trim 1 inch on a base of draught, also one giving the longitudinal position of the centre of buoyancy. Then at the original displacement if the trim be by the stem, the distance of G abaft B is equal to the trim in inches × moment to change trim 1 inch at that draught (found from the curve) ÷ displacement. Knowing the longitudinal position of B from the curve, that of G is obtained. The change in this due to the added weights is then determined; and the above process, reversed and using the final moment, positions of B and G, and displacement, gives the final trim.

Examples of the above methods are given in the inclining experiment described below.

To Determine the Vertical Position of a Ship's Centre of Gravity by Experiment.

Fig. 134.



In fig. 134 let MZ be the upright axis of a ship; her centre of gravity then lies somewhere in that axis. M is the metacentre, and GM its vertical height above the centre of gravity G.

If a weight P be moved transversely through a distance $PP' = d$, it will heel the vessel over through an angle θ, and her centre of gravity will then shift in a direction GG′ parallel to that in which the centre of gravity of the weight has been shifted. Let MT be parallel to GG′ and TG′ parallel to GM; let P = weight shifted in tons, and W = displacement of ship in tons: then

$$MT = GG' = \frac{P \times d}{W}, \text{ and } GM = GG' \operatorname{cotan} \theta = \frac{P \times d}{W} \operatorname{cotan} \theta.$$

In practice the ballast is usually in the form of pig iron arranged in two parallel rows on the port and starboard sides of the upper deck.

The method of performing the experiment is illustrated by the calculations below, which correspond to a light cruiser, whose metacentric diagram is given in fig. 121.